Radiative Corrections to π-e Scattering*†

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The radiative corrections to order α^3 for π-e scattering have been calculated. The inelastic part of the radiative corrections have been realistically handled assuming an experimental setup, much like that of Fitch, Leipuner *et al.* at Brookhaven, where the energies of both final particles are measured. Although large log-squared terms are found in the results, they are always cancelled by comparable terms. The reason for their appearance can be directly related to the nature of the experimental situation.

I. INTRODUCTION

At Brookhaven,[1] experiments designed to measure the electromagnetic structure effects the π^- meson have recently been performed. Electrons scattered out of a target by high-energy (20–25 BeV) π^- mesons are momentum analyzed and counted in coincidence with the scattered π^- mesons. The cross section thus obtained when compared with the theoretically calculated cross section, assuming the π^- meson to be a point source, yields information on the structure of the π^- meson, i.e., its electromagnetic form factor. The purpose of this paper is to calculate the quantum electrodynamic parts of the radiative corrections for this process.

This calculation can be separated into two parts: elastic and inelastic. Elastic parts refer to those Feynman diagrams which have a final electron and a final pion, but no final photons. Inelastic parts refer to those diagrams in which, in addition to the final electron and pion, a photon is also emitted. Since the final states of elastic and inelastic diagrams are different, no interference between them can occur and hence the observable cross section is simply the sum of the elastic and inelastic cross sections.

Once the rules for forming the matrix elements corresponding to the Feynman diagrams are given,[2] the evaluation of the elastic cross section is straightforward. The usual renormalization techniques[3] are applicable for the removal of the so-called ultraviolet divergencies. The infrared divergencies are avoided (in the inelastic case as well) by assuming a small photon mass λ whenever necessary. Because of energy-momentum conservation expressed by the appearance of the δ function in the final-state integration, this integration is trivial.

Not so for the inelastic cross section. Since the final state here has an additional photon, if we again absorb the δ function in performing the integrations over the electron and pion variables, we are yet left to do a final-state integration over the photon variables. In addition to the complications of this integration *per se*, we have also to determine the region of integration as limited by the experimental conditions.

Let E', $\mathcal{E}'$ be the energies of the final particles and let θ', θ be their scattering angles, respectively. Clearly, for elastic scattering, the condition of energy-momentum conservation gives three relations for these four quantities and consequently only one is independent. In most scattering experiments, one measures, with imprecisions, any two of these four quantities and admits only those events which agree, within the limitations of these imprecisions, with the elastic condition of energy-momentum conservation. That is to say, the experiment relates (albeit loosely) the quantities it measures.

These scattering experiments can then be divided into the following three categories:

(a) E' and θ' or $\mathcal{E}'$ and θ are measured, i.e., only one particle is detected;
(b) θ' and θ are measured;
(c) E' and $\mathcal{E}'$ are measured.

The technique for handling case (a) depends on whether the heavier or the lighter of the particles is detected[4] and further depends on which of the angular or the energy imprecisions is the larger when properly compared. Meister and Yennie[5] have given a complete treatment of case (a) and Tsai[6] has treated case (b). Case (c) is the one pertaining to our problem and in this sense this paper is to be considered a completion of the aforementioned works.

The main task in performing any radiative correction calculation is then to properly transform the relation imposed by the experiment (on the measured quantities) to a restriction on the region of integration of the final photon variables.[7] We refer to this region as the experi-

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[1] These experiments have been carried out by M. Barton, D. Cassell, R. Crittendem, V. L. Fitch, and L. Leipuner.
[2] Cf., Silvan S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row Peterson and Company, Evanston, Illinois, 1961), Sec. 14.c, p. 483, for scalar electrodynamics.
[3] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).
[4] A. S. Krass, Phys. Rev. **125**, 2172 (1962); Y. S. Tsai, *ibid.* **122**, 1898 (1961).
[5] N. Meister and D. R. Yennie, Phys. Rev. **130**, 1210 (1963).
[6] Y. S. Tsai, Phys. Rev. **120**, 269 (1960).
[7] As is pointed out by Tsai in Ref. 6, the radiative corrections calculated by M. L. G. Redhead, Proc. Roy. Soc. (London) **A220**, 219 (1953) and R. V. Polovin, Zh. Eksperim. i Teor. Fiz. **31**, 449 (1956) [English transl.: Soviet Phys.—JETP **4**, 385 (1957)], are deficient because the experimental conditions are handled unrealistically.

mentally restricted region. This transformation is always done via the four equations which express energy-momentum conservation for the three final particles. These equations themselves restrict the region of photon integration and we refer to this region as the kinematically restricted region. In cases (a) and (b), since the experimental relationship involves angles, the transformation to the experimentally restricted region makes use of all four energy-momentum conservation equations. Accordingly, the information contained in these equations is already implicit in the resulting experimentally restricted region. That is, for cases (a) and (b), the experimentally restricted region is completely contained in the kinematically restricted region.

For case (c) the situation is quite different. Since only energies are measured, in transforming to the experimentally restricted region the momentum conservation equations remain untouched and only the energy equation, viz.,

$$E'+\mathcal{E}'+\omega=\text{constant}, \tag{I.1}$$

where ω is the photon energy, is needed. Consequently, for case (c), the kinematically restricted region has yet to be considered and the intersection of this with the experimentally resticted region gives the allowed photon region.

From the linearity of Eq. (I.1) it can be seen that the experimentally restricted region is simply the isotropic region given by $\omega\leq\Delta E$, where ΔE is a measure of the experimental imprecisions, about which more will be said later. The kinematically resticted region, on the other hand, is more difficult to arrive at and will be discussed in detail in Sec. III.

The notation used in this paper is nearly identical with that of Jauch and Rohrlich.[8] The units used are $\hbar=c=1$ and $e^2/4\pi=\alpha$. The relativistic notation used is such that

$$a_\nu\equiv(\mathbf{a},a_0)\,,\quad a_\nu\gamma^\nu\equiv \boldsymbol{a}\,,$$
$$a_\nu b^\nu\equiv a\cdot b\equiv\mathbf{a}\cdot\mathbf{b}-a_0b_0\,.$$

The γ_ν are 4×4 matrices satisfying the relation

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}\,,$$

where

$$g_{11}=g_{22}=g_{33}=-g_{00}=1$$

and all other g_{ij}'s are zero. p and q refer to the four momenta associated with the incoming electron and pion, respectively, and primed quantities refer to outgoing variables. In the lab system we have

$$p=(0,m)\,,\quad q=(\mathbf{q},E)\,,\quad p'=(\mathbf{p}',\mathcal{E}')\,,\quad q'=(\mathbf{q}',E')\,.$$

The following definitions are constant throughout this paper:

$$p_3=p-p'\,,\quad q_3=q'-q\,,\quad (p_3=q_3\,,\text{ for elastic case})$$
$$\kappa=p_3{}^2/2m^2\,,\quad \kappa'=p_3{}^2/2\mu^2\,,$$

[8] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Cambridge, Massachusetts, 1955).

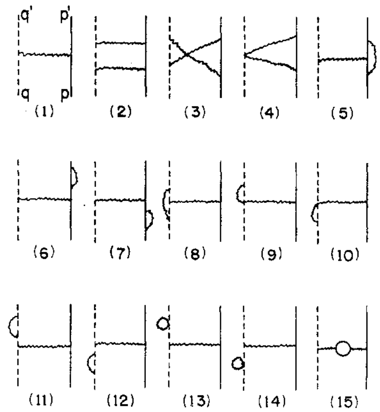

FIG. 1. Feynman diagrams, to order e^4, for elastic scattering.

where μ is the mass of the π^- meson, and m the mass of the electron.

The symbol $\doteq$ is defined by $A\doteq B$ if and only if

$$\bar{u}(\mathbf{p}')Au(\mathbf{p})=\bar{u}(\mathbf{p}')Bu(\mathbf{p})\,,$$

where $u(\mathbf{p})$ is the electron spinor.

Terms like $(\alpha/\pi)\log^2(2\mathcal{E}'/m)$ in the radiative corrections (which are considered undesirable because they cast aspersions on the validity of the perturbation expansion) have been shown by Tsai[6] to completely cancel for his case when the experimental conditions are realistically handled. It would appear that our final result does contain such a term but this is illusory. For a more careful examination of our final result also reveals the term $-(\alpha/\pi)\log^2(\Delta E/\bar{\omega})$, where $\bar{\omega}$ is defined in Eq. (III.13). In Sec. IV we shall show that this latter term should always be grouped with $(\alpha/\pi)\log^2(2\mathcal{E}'/m)$, and the two terms tend to cancel.

It will be seen that this result arises precisely because of the peculiarity of our allowed photon region, i.e., because in our case it is possible to distinguish between the experimentally and kinematically restricted regions. The first term can then be viewed as coming from the isotropic experimentally restricted region and the second term compensates for this overestimate. Because in cases (a) and (b) the two regions blend and are indistinguishable this compensation (cancellation) takes place implicitly.

II. ELASTIC SCATTERING

The Feynman diagrams, to order e^4, for the scattering of electrons by pions are depicted in Fig. 1.

Let M_i represent the matrix element corresponding to diagram (i) of Fig. 1. Then the lowest order matrix element (M_1) is given by

$$M_1=-cm\bar{u}(\mathbf{p}')Qu(\mathbf{p})\,, \tag{II.1}$$

where

$$c=\left(\frac{\alpha}{2\pi}\right)\frac{1}{p_3{}^2(E\mathcal{E}E'\mathcal{E}')^{1/2}}\,. \tag{II.2}$$

The elastic scattering cross section $d\sigma_{\rm el}$, is given by[9]

$$d\sigma_{\rm el}=\frac{(2\pi)^2E\mathcal{E}}{2[(pq)^2-m^2\mu^2]^{1/2}}\times\int\int d^3p'd^3q'\delta^4(p+q-p'-q')\sum_{\rm spins}M^\dagger M\,, \quad \text{(II.3)}$$

M is the sum of the matrix elements M_i, i.e.,

$$M=M_1+\sum_{i=2}^{15}M_i. \quad \text{(II.4)}$$

Then to order α^3

$$M^\dagger M=M_1{}^\dagger M_1+2\,{\rm Re}\sum_{i=2}^{15}M_1{}^\dagger M_i. \quad \text{(II.5)}$$

The first term $M_1{}^\dagger M_1$ of Eq. (II.5) gives rise to the lowest order contribution to $d\sigma_{\rm el}$ and the remaining terms are the radiative corrections to order α^3. From Eq. (II.1) we get

$$\sum_{\rm spins}M_1{}^\dagger M_1=c^2m^2\sum_{\rm spins}[\bar{u}(\mathbf{p}')Qu(\mathbf{p})]^*[\bar{u}(\mathbf{p}')Qu(\mathbf{p})]$$
$$=-\tfrac{1}{4}c^2\,{\rm Tr}[(ip-m)Q(ip'-m)Q]=c^2T_0 \quad \text{(II.6)}$$

and

$$T_0=2[4(pq)(pq')-p_3{}^2\mu^2]. \quad \text{(II.7)}$$

Let $d\sigma_0$ be the scattering cross section to lowest order. Making use of Eq. (II.2) for c we have

$$d\sigma_0=\frac{\alpha^2}{2[(pq)^2-(m\mu)^2]^{1/2}}\times\int\frac{d^3p'}{\mathcal{E}'}\int\frac{d^3q'T_0}{E'(p_3{}^2)^2}\delta(p+q-p'-q'). \quad \text{(II.8)}$$

With the aid of the space part of the δ function the d^3q' integration in Eq. (II.8) is trivial. There then remains to do the integration over $d^3p'=p'\mathcal{E}'d\Omega_{p'}d\mathcal{E}'$. Usually the remaining portion of the δ function is used to perform the $d\mathcal{E}'$ integration and what results is an expression for $d\sigma/d\Omega$, i.e., cross section per unit sold angle. However, in view of what the experiment actually measures, the more relevant thing to do is to use the remaining portion of the δ function together with the condition of axial symmetry and render the $d\Omega_{p'}$ integration. This then will lead to an expression for $d\sigma/d\mathcal{E}'$, the cross section per unit (scattered electron) energy, which is, in fact, what is measured. Thus, for the lab system, we get

$$d\sigma_0/d\mathcal{E}'=\pi\alpha^2T_0/m\mathbf{q}^2(p_3{}^2)^2. \quad \text{(II.9)}$$

The matrix elements arising from diagrams 5–15 of Fig. 1 are divergent for large values of the momenta of the virtual photons. These divergencies can be removed by means of the usual mass and charge renormalization procedure.[3] Let us define[10]

$$K(p_a,p_b)=(p_ap_b)\int_0^1\frac{dy}{p_y{}^2}\ln\left(\frac{p_y{}^2}{\lambda^2}\right) \quad \text{(II.10)}$$

and

$$\mu(p_a,p_b)=\int_0^1\frac{dy}{p_y{}^2}, \quad \text{(II.11)}$$

where $p_y=yp_a+(1-y)p_b$. $K(p_a,p_b)$ contains the infrared divergence through the appearance of λ, the fictitious photon mass.

In terms of $K(p_a,p_b)$ and $\mu(p_a,p_b)$ we may write

$$M_{5-7}=(-\alpha/2\pi)[K(p,p')-K(p,p)-(\tfrac{1}{4}p_3{}^2+2(pp'))\mu(p,p')+2]M_1, \quad \text{(II.12)}$$

$$M_{15}=(-\alpha/2\pi)[(10/9)-\tfrac{1}{3}(p+p')^2\mu(p,p')]M_1, \quad \text{(II.13)}$$

$$M_{8-14}=(-\alpha/2\pi)[K(q,q')-K(q,q)-2(qq')\mu(qq')+2]M_1. \quad \text{(II.14)}$$

Equation (II.12) comes from the renormalized electron vertex diagrams, and Eq. (II.13) from the renormalized electron vacuum-polarization diagram.[11] Equation (II.14) comes from the renormalized meson vertex diagrams, and a meson vacuum-polarization diagram should logically be included in Fig. 1. However, after renormalization this diagram contributes a negligible amount and has therefore been omitted.[12]

Unfortunately, M_2, M_3, and M_4 cannot be rendered into so simple a form as some factor times M_1. These two-photon exchange diagrams offer the greatest computational difficulties. By applying the rules of correspondence we get

$$M_i=-cm\bar{u}(\mathbf{p}')p_3{}^2J_iu(\mathbf{p}),\quad (i=2,3,4) \quad \text{(II.15)}$$

where

$$J_2=\frac{-e^2}{(2\pi)^4}\int\frac{(Q+k)(i(p-k)-m)(2q+k)d^4k}{(k^2-2pk)(k^2+2qk)(k^2+\lambda^2)[(k-p_3)^2+\lambda^2]}, \quad \text{(II.16)}$$

$$J_3=\frac{-e^2}{(2\pi)^4}\int\frac{(Q-k)(i(p-k)-m)(2q'-k)d^4k}{(k^2-2pk)(k^2-2q'k)(k^2+\lambda^2)[(k-p_3)^2+\lambda^2]}, \quad \text{(II.17)}$$

$$J_4=\frac{-4e^2}{(2\pi)^4}\int\frac{(m-ik)d^4k}{(k^2-2pk)(k-p_3)^2k^2}. \quad \text{(II.18)}$$

J_2 and J_3 contain infrared divergencies and for that

[9] Reference 8, Eq. (8–49).

[10] This notation for the infrared terms, $K(p_a,p_b)$, is identical with that of Tsai in Ref. 4.

[11] Compare Eqs. (II.12) and (II.13) with Eqs. (II.5) and (II.4), respectively, of Tsai, Ref. 4, where terms of order $m/\mathcal{E}'$ have been neglected compared with unity.

[12] Cf., Appendix B for the derivation of Eq. (II.14) and the meson vacuum-polarization term.

reason we have included a fictitious photon mass λ in these integrals. The infrared divergencies can be separated out; i.e., we can write J_2 and J_3 each as a sum of two terms, one term containing the infrared divergency, and the other term free of divergencies. Such a separation is clearly not unique. For definiteness we will separate by extracting the infrared term using the technique developed by Yennie, Frautschi, and Suura.[13]

Let us consider J_3, Eq. (II.17). When either of the four-momenta of the photon propagators approaches zero, i.e., $k\to 0$ or $k-p_3\to 0$, we have infrared divergence. Suppose $k\to 0$, then the infrared contribution from J_3 due to $k\to 0$, is obtained by neglecting k in the numerator and in $(k-p_3)^2$. We then obtain

$$J_3^{\lambda'}\doteq-\frac{4ie^2}{(2\pi)^4}\frac{Q}{p_3^2}\int\frac{(pq')d^4k}{(k^2-2pk)(k^2-2q'\cdot k)(k^2+\lambda^2)}=\frac{-\alpha}{2\pi}\frac{Q}{p_3^2}K(p,q'),$$

where $K(p_a,p_b)$ is defined in Eq. (II.10).

Similarly the infrared contribution from J_3 due to $k-p_3\to 0$ can be obtained by a substitution $k-p_3\to k$ in J_3, and we have

$$J_3^{\lambda''}\doteq(-\alpha/2\pi)(Q/p_3^2)K(p',q).$$

Thus we have for the extracted infrared term

$$J_3^{\lambda}=J_3^{\lambda'}+J_3^{\lambda''}\doteq(-\alpha/2\pi)(Q/p_3^2)\times[K(p,q')+K(p',q)]. \quad \text{(II.19)}$$

We can then write

$$J_3=J_3^{\lambda}+J_3^{0} \quad \text{(II.20)}$$

and J_3^0 is nondivergent. The matrix element M_3^λ, arising from J_3^λ now has the particularly simple form

$$M_3^{\lambda}=-(\alpha/2\pi)[K(p,q')+K(p',q)]M_1. \quad \text{(II.21)}$$

Analogously for J_2 we may write

$$J_2=J_2^{\lambda}+J_2^{0}, \quad \text{(II.22)}$$

where

$$J_2^{\lambda}\doteq(\alpha/2\pi)(Q/p_3^2)[K(p,-q)+K(p',-q')] \quad \text{(II.23)}$$

and J_2^0 is nondivergent.

$K(p_a,-p_b)$ is complex. This added complication in M_2 arises from the fact that the intermediate state can become real; thus we have to cross a pole in the path of integration with respect to k, the photon four-momentum. The path of integration around the pole is taken care of by giving to m, μ, and λ small negative imaginary parts. In our calculation only the real part of $K(p_a,-p_b)$ contributes to the cross section. Hence we write[14]

$$\mathrm{Re}[K(p,-q)+K(p',-q')]=K(p,q)+K(p',q')+\delta_k, \quad \text{(II.24)}$$

(which defines δ_k) and therefore the matrix element arising from J_2^λ is

$$M_2^{\lambda}=(\alpha/2\pi)[K(p,q)+K(p',q')+\delta_k]M_1. \quad \text{(II.25)}$$

Finally, if we let $(d\sigma_0/d\mathcal{E}')(\alpha/\pi)\delta_J$ be the contribution to the cross section arising from J_2^0, J_3^0, and J_4, we get for the elastic cross section

$$\frac{d\sigma_{\mathrm{el}}}{d\mathcal{E}'}=\frac{d\sigma_0}{d\mathcal{E}'}\left[1+\frac{\alpha}{\pi}(\delta_1+\delta_2+\delta_\lambda)\right], \quad \text{(II.26)}$$

where

$$\delta_\lambda=K(p,p)+K(q,q)-K(p',q)-K(p,q')-K(p,p')-K(q,q')+K(p,q)+K(p',q'), \quad \text{(II.27)}$$

$$\delta_2=\delta_K+\delta_J, \quad \text{(II.28)}$$

and neglecting $m/\mathcal{E}'$ compared with unity

$$\delta_1=-(13/12)p_3^2\mu(p,p')+2(qq')\mu(q,q')-(46/9). \quad \text{(II.29)}$$

δ_λ contains all the infrared terms and will completely cancel when added to the inelastic cross section. δ_1 is the usual radiative corrections due to the vertex and vacuum polarization diagrams. By means of Eq. (A48) it can be put into the more calculable form

$$\delta_1=\frac{13}{6}\ln\frac{p_3^2}{m^2}+\frac{2(\kappa'+1)}{a'}\ln\frac{a'+\kappa'}{a'-\kappa'}-\frac{46}{9}, \quad \text{(II.30)}$$

where $\kappa'=p_3^2/2\mu^2$, $a'=(\kappa'^2+2\kappa')^{1/2}$.

Finally, it is shown in Appendix C that in terms of the Spence function $\Phi(x)$, defined in Eq. (A49), we get

$$\delta_2=2\left[\ln\left(\frac{4(pq)(pq')}{\mu^2p_3^2}\right)\ln\left(\frac{pq'}{pq}\right)+\Phi\left(\frac{\mu^2+2pq'}{\mu^2}\right)-\Phi\left(\frac{\mu^2+2pq}{\mu^2}\right)\right]$$
$$+\frac{p_3^2(\mu^2+2pq')}{T_0}\left[\tfrac{1}{2}\ln^2\left(\frac{(2pq')^2}{\mu^2p_3^2}\right)+2\Phi\left(\frac{\mu^2+2(pq')}{\mu^2}\right)+\frac{2\pi^2}{3}\right]-\frac{p_3^2(\mu^2-2pq)}{T_0}\left[\tfrac{1}{2}\ln^2\left(\frac{(2pq)^2}{\mu^2p_3^2}\right)+2\Phi\left(\frac{\mu^2-2pq}{\mu^2}\right)-\frac{4\pi^2}{3}\right]$$
$$-\frac{2p_3^2(pQ)}{T_0}\left(\frac{\kappa'+1}{a'}\right)\left[\ln 2\kappa'\ln\frac{a'-\kappa'}{a'+\kappa'}+\Phi(a'-\kappa')-\Phi(-(a'+\kappa'))-\pi^2\right]. \quad \text{(II.31)}$$

We have used Eq. (A45) for δ_K to derive Eq. (II.31).

[13] D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) 13, 379 (1961).
[14] Cf., Appendix A, Sec. D, Eqs. (A45) and (A46).

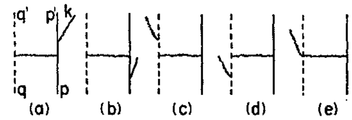

FIG. 2. Feynman diagrams, to order e^3, for inelastic scattering.

III. INELASTIC SCATTERING

The Feynman diagrams, to order e^3, associated with inelastic π^--e scattering are shown in Fig. 2. The matrix element corresponding to these diagrams is given by

$$M=\frac{-e^3}{(2\pi)^{7/2}}\frac{m}{2(2\omega\mathcal{E}\mathcal{E}'EE')^{1/2}}\bar{u}(\mathbf{p}')(J\cdot\epsilon)u(\mathbf{p}), \quad \text{(III.1)}$$

where

$$J^\nu\epsilon_\nu=\frac{1}{(p_3-k)^2}\left\{\left[\frac{p\cdot\epsilon}{p\cdot k}-\frac{p'\cdot\epsilon}{p'\cdot k}\right]Q-\frac{1}{2}\left[\frac{Qk\varepsilon}{p\cdot k}+\frac{\varepsilon kQ}{p'\cdot k}\right]\right\}+\frac{1}{p_3{}^2}\left\{\left[\frac{q\cdot\epsilon}{q\cdot k}-\frac{q'\cdot\epsilon}{q'\cdot k}\right]Q-\left(\frac{q\cdot\epsilon}{q\cdot k}+\frac{q'\cdot\epsilon}{q'\cdot k}\right)k+2\varepsilon\right\}. \quad \text{(III.2)}$$

ϵ is the polarization four-vector of the emitted photon; hence $k\cdot\epsilon=0$. This result, as well as the appearance of $J\cdot\epsilon$ between the spinors $\bar{u}$ and u, have been used in deriving Eq. (III.2). Note that Eq. (III.2) satisfies the condition of gauge invariance, i.e., $J\cdot k=0$.

The scattering cross section for the inelastic processes $d\sigma_{\text{inel}}$, is given by

$$d\sigma_{\text{inel}}=\frac{(2\pi)^2}{2}\frac{E\mathcal{E}}{[(pq)^2-m^2\mu^2]^{1/2}}\int\int\int d^3k d^3p' d^3q' \times\delta^4(p+q-p'-q'-k)\sum_{\text{spins}}M^\dagger M, \quad \text{(III.3)}$$

where $\sum_{\text{spins}}$ is the summation over electron spins as well as photon polarizations. Using expression (III.1) for M and employing the usual techniques for spin summations, we can write

$$d\sigma_{\text{inel}}=\frac{\alpha^3}{(2\pi)^2}\frac{1}{[(pq)^2-m^2\mu^2]^{1/2}}\int\frac{d^3k}{2\omega}\int\frac{d^3p'}{\mathcal{E}'}\int\frac{d^3q'}{E'} \times\delta^4(p+q-p'-q'-k)A, \quad \text{(III.4)}$$

where

$$A=\tfrac{1}{4}\,\mathrm{Tr}[(ip-m)\bar{J}_\nu(ip'-m)J^\nu]. \quad \text{(III.5)}$$

The space part of the δ function allows us to do the d^3q' integration in Eq. (III.4). This gives

$$d\sigma_{\text{inel}}=\frac{\alpha^3}{(2\pi)^2}\frac{1}{[(pq)^2-m^2\mu^2]^{1/2}}\int\frac{d^3k}{2\omega}\int\frac{d^3p'}{\mathcal{E}'E'} \times\delta(\omega+E'-t_0)A_{\mathbf{q}'=\mathbf{t}-\mathbf{k}},$$

where $t=p+q-p'$ and $E'=[\mu^2+(\mathbf{t}-\mathbf{k})^2]^{1/2}$. If, as shown in Fig. 3, we choose our coordinate system such that the z axis is along $\mathbf{q}-\mathbf{k}$ and such that the $x-z$ plane contains $\mathbf{q}$ and $\mathbf{k}$, then we may write $d^3p'/\mathcal{E}'=|\mathbf{p}'|d\mathcal{E}'d\varphi d(\cos\theta)$. θ is the angle between $\mathbf{p}'$ and $\mathbf{q}-\mathbf{k}$. As in the elastic case, we now employ the remaining part of the δ function to perform the $d(\cos\theta)$ integration. In the lab system we have

$$dE'/d(\cos\theta)=|\mathbf{p}'||\mathbf{q}-\mathbf{k}|/E',$$

and, therefore,

$$\frac{d\sigma_{\text{inel}}}{d\mathcal{E}'}=\frac{\alpha^3}{(2\pi)^2}\frac{1}{m|\mathbf{q}|}\int\frac{d^3k}{2\omega|\mathbf{q}-\mathbf{k}|} \times\int_0^{2\pi}d\varphi A\Big|_{\cos\theta=c(\mathbf{k})}^{\mathbf{q}'=\mathbf{t}-\mathbf{k}}. \quad \text{(III.6)}$$

$A\Big|_{\cos\theta=c(\mathbf{k})}^{\mathbf{q}'=\mathbf{t}-\mathbf{k}}$ is meant to represent A of Eq. (III.5) evaluated at $\mathbf{q}=\mathbf{t}-\mathbf{k}$ and $\cos\theta=c(\mathbf{k})$, where $(\mathbf{k})$ is gotten from the condition $\omega+E'-t_0=0$, i.e.,

$$\cos\theta=\frac{(\mathcal{E}'-m)(E+m)+\omega(E+m-\mathcal{E}'-|\mathbf{q}|\cos\phi)}{|\mathbf{p}'||\mathbf{q}-\mathbf{k}|}\equiv c(\mathbf{k}). \quad \text{(III.7)}$$

We have let ϕ be the angle between $\mathbf{q}$ and $\mathbf{k}$.

The equality $c(\mathbf{k})=\cos\theta$, imposes a restriction on the region of integration of d^3k. That is, only those photons of four-momentum k for which $|c(\mathbf{k})|\leq 1$ are kinematically allowed. Hence the boundary of the integration region as determined by this kinematic restriction is gotten by solving $[c(\mathbf{k})]^2=1$ or equivalently,

$$a\omega_k{}^2+2b\omega_k+c=0, \quad \text{(III.8)}$$

where

$$\begin{aligned} a&=(E+m-\mathcal{E}'-x)^2-\mathbf{p}'^2,\\ b&=(E+m-\mathcal{E}'-x)(\mathcal{E}'-m)(E+m)+\mathbf{p}'^2x,\\ c&=(\mathcal{E}'-m)^2(E+m)^2-\mathbf{p}'^2\mathbf{q}^2,\\ x&=|\mathbf{q}|\cos\phi. \end{aligned} \quad \text{(III.9)}$$

So far we have in no way taken account of any limitations imposed on k by the experiment. That is, in the final-state integration we have allowed all possible three-body states limited only by the condition $p'+q'+k$ = constant four-vector. The experiment however, in not counting as events those final states which fail, within

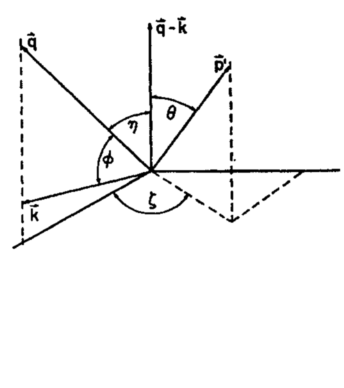

FIG. 3. The coordinate geometry for the inelastic final-state integration.

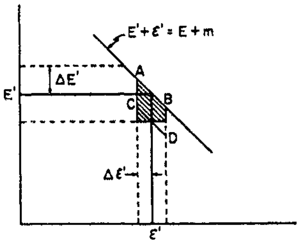

FIG. 4. In the E', $\mathcal{E}'$ plane, the shaded area represents those events allowed by the experimental restriction. We approximate this shaded area by the parallelogram $ABDC$.

the accuracy of the detectors, to be "elastic," further relates p', q', and k. The experimental situation is taken to be the following: the final energies of both the pion and the electron are measured. Only those events for which the pion energy is $E'\pm\Delta E'$ and the electron energy is $\mathcal{E}'\pm\Delta\mathcal{E}'$ such that $E'+\mathcal{E}'=E+m$, are counted. $\Delta E'$ and $\Delta\mathcal{E}'$ are the imprecisions in the measurements of these energies. In Fig. 4 we plot E' versus $\mathcal{E}'$. The shaded area corresponds to those events counted by our experiment. This relationship between E' and $\mathcal{E}'$, we now transform into a restriction on the possible values of **k**. It is clear that since no angles are measured, the experiment in no way restricts the orientation of **k**. We therefore get for the experimental restriction the isotropic condition

$$\omega\leq\Delta E\,, \tag{III.10}$$

where $\Delta E=\Delta E'+\Delta\mathcal{E}'$. We shall elaborate upon this later.

Let us now return to finding the solution of Eq. (III.8); that is, the boundary of the kinematically allowed region. It is clear that when $E-x\gg m$, then $b^2\gg ac$ and

$$\omega_k\simeq\frac{-c}{2b}=\frac{2mE(E-\mathcal{E}')-\mu^2\mathcal{E}'}{2(E-\mathcal{E}')(E-|\mathbf{q}|\cos\phi)}\,. \tag{III.11}$$

We have approximated b and c of Eq. (III.9) by their leading terms, assuming $E-x\gg m$. Equation (III.11) is the equation of an ellipse and except for the region $\cos\phi\approx1$ it gives to a high degree of accuracy the boundary for the kinematically restricted region. For $E-x\approx m$, Eq. (III.11) is no longer a good approximation for the solution of Eq. (III.8). To obtain the kinematically restricted region for $\cos\phi\approx 1$ we need to solve the quadratic exactly. Fortunately we can avoid this chore by considering the experimental restriction $\omega\leq\Delta E$. Since we are interested in the allowed region of **k**, i.e., the intersection of the kinematically restricted and experimentally restricted region, then, as shown in Fig. 5 whenever $\Delta E<\omega_k$ we need not specifically know ω_k. Therefore, in order that the approximate form of ω_k (III.11) be sufficient, we want the condition $E-x\gg m$ should still obtain at $\Delta E=\omega_k$, i.e.,

$$\Delta E\ll\frac{2mE(E-\mathcal{E}')-\mu^2\mathcal{E}'}{2m(E-\mathcal{E}')}\,. \tag{III.12}$$

We rewrite this as[15]

$$\Delta E\ll[(\mu^2+2mE)/2m(E-\mathcal{E}')](\mathcal{E}_{\max}'-\mathcal{E}')\,,$$

where $\mathcal{E}_{\max}'\simeq 2mE^2/(\mu^2+2mE)$ is the maximum possible value of $\mathcal{E}'$. This says that if we get too near maximum momentum transfer, the phase space available to the emitted photon becomes more inhibitive than the experimental restriction. In such an event the experiment in no way discriminates between elastic and inelastic processes.

Actually since the main contribution to the final-state integration comes from small values of ω, our results aren't too sensitive to errors in the shape of ω_k for $E-x\approx m$. Hence, the range of validity of our results will be wider than what is implied by Eq. (III.12). We therefore take as the integration region for d^3k the shaded area of Fig. 5.

Let $\bar{\omega}$ be the distance from the focus to the vertex of the ellipse given by Eq. (III.11), i.e.,

$$\bar{\omega}=\frac{2mE(E-\mathcal{E}')-\mu^2\mathcal{E}'}{2(E-\mathcal{E}')(E-|\mathbf{q}|)}=\frac{T_0}{16mE(E-\mathcal{E}')}\,. \tag{III.13}$$

We shall call those photons with energy $\omega<\bar{\omega}$ "soft photons" and those with $\omega>\bar{\omega}$ "hard photons." Then $d\sigma_{\text{inel}}=d\sigma^s+d\sigma^h$, where $d\sigma^s$ is the cross section arising from the emission of soft photons and $d\sigma^h$ from hard photons. By Eq. (III.6) we may write

$$\frac{d\sigma^h}{d\mathcal{E}'}=\frac{\alpha^3}{2\pi}\frac{1}{2m|\mathbf{q}|^2}\int_{\bar{\omega}}^{\Delta E}\omega d\omega \times\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{[\mathbf{q}^2+\omega^2-2\omega E+2\omega y]^{1/2}}\int_0^{2\pi}Ad\varphi\,, \tag{III.14}$$

where

$$\bar{y}=\frac{2mE(E-\mathcal{E}')-\mu^2\mathcal{E}'}{2\omega(E-\mathcal{E}')}=\frac{T_0}{8m\omega(E-\mathcal{E}')}=\frac{2E\bar{\omega}}{\omega}\,. \tag{III.15}$$

In deriving this result we have first written $d^3k=2\pi\omega^2d\omega d(\cos\phi)$ and then let $y=E-|\mathbf{q}|\cos\phi$. Strictly speaking, Eq. (III.14) is not quite correct. It weights photons of different energies equally. However, photons with energy $\omega>\Delta E'-\Delta\mathcal{E}'$ should be weighted less than photons with energy $\omega<\Delta E'-\Delta\mathcal{E}'$. Specifically, one

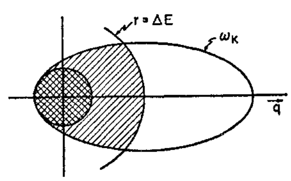

FIG. 5. The phase space available to the emitted photon is represented by the shaded area. The cross-hatched area represents the soft photon region.

[15] Actually, if we solve Eq. (III.8) for $\phi=0$, we get $\omega_k=\mathcal{E}_{\max}'-\mathcal{E}'$. Hence, when $\Delta E=\mathcal{E}_{\max}'-\mathcal{E}'$ the kinematically restricted region is wholly contained in the experimentally restricted region.

should include in the above integral the weight function $g(\omega)$ given by

$$g(\omega)=1 \quad \text{for} \quad \bar{\omega}<\omega<\Delta E'-\Delta\mathcal{E}'$$
$$=\frac{\Delta E'+\Delta\mathcal{E}'-\omega}{2\Delta\mathcal{E}'} \quad \text{for} \quad \Delta E'-\mathcal{E}\Delta'<\omega<\Delta E'+\Delta\mathcal{E}'.$$

This can be seen from the following argument: All the events corresponding to the points in the shaded area of Fig. 4 have an equal likelihood of occurring. Let p be any point in this area and let d be its distance to the line $E'+\mathcal{E}'=E+m$. Then if ω_p is the energy of the photon produced in the event corresponding to p, we must have $\omega_p=\sqrt{2}d$. That is, all photons of energy ω_p must come from an event represented by one of the points on the line segment parallel to the line $E'+\mathcal{E}'=E+m$, passing through p and contained within the shaded area. Hence, the probability of having ω_p is proportional to the length of this line segment. Thus, $g(\omega)$ above.

However, since the main contribution to the integral comes from small ω, and if we assume $\Delta\mathcal{E}'\ll\Delta E'$, we can safely approximate the effect of $g(\omega)$ by taking $\Delta E=\Delta E'$. That is, replacing the shaded area of Fig. 4 by the parallelogram $ABDC$.

As we have already indicated, the error introduced by relaxing condition (III.12) is negligible. Hence one can expect (III.14) to apply even when $\Delta E>\mathcal{E}_{\max}'-\mathcal{E}'$. However, in such an event we first need modify (III.14) by replacing ΔE by $\mathcal{E}_{\max}'-\mathcal{E}'$, as can be seen from footnote 15.

A. Soft Photons

The region of **k** integration for soft photons is depicted by the cross-hatched area in Fig. 5. It is the largest possible isotropic region. Since $\bar{\omega}$ (III.13) is smaller than $m/2$ we can neglect k in the δ function of Eq. (III.4), and rewrite Eq. (III.2) as

$$J^\nu\epsilon_\nu=\frac{Q}{p_3{}^2}\left[\frac{p\cdot\epsilon}{p\cdot k}-\frac{p'\cdot\epsilon}{p'\cdot k}+\frac{q\cdot\epsilon}{q\cdot k}-\frac{q'\cdot\epsilon}{q'\cdot k}\right].$$

Then if we compare Eq. (III.4) with Eq. (II.8) it can be seen that

$$d\sigma^s=(\alpha/\pi)d\sigma_0\delta_s\,, \tag{III.16}$$

where

$$\delta_s=\frac{1}{4\pi}\int_0^{\bar{\omega}}\frac{k^2dk}{[k^2+\lambda^2]^{1/2}}\int d\Omega_k\left[\frac{p}{p\cdot k}-\frac{p'}{p'\cdot k}+\frac{q}{q\cdot k}-\frac{q'}{q'\cdot k}\right]^2.$$

In order to avoid the infrared divergence which occurs in the soft photon cross section, we have assumed a small photon mass λ. We can write δ_s as

$$\delta_s=-\delta_\lambda+\sum_{a\leq b}\epsilon_{ab}I(p_a,p_b)\,, \tag{III.17}$$

where we have anticipated the form of the infrared terms by explicitly separating δ_λ given in Eq. (II.27). $\epsilon_{aa}=1$ and for $a\neq b$, $\epsilon_{ab}=\pm 2$ according as p_a and p_b are alike or unalike with respect to the attribute of being an incoming or outgoing momentum. $I(p_a,p_b)$ is now divergentless and is given by

$$I(p_a,p_b)=\frac{1}{4\pi}\int_0^{\omega}\frac{k^2dk}{[k^2+\lambda^2]^{1/2}}\int\frac{(p_ap_b)d\Omega_k}{(p_ak)(p_bk)}+\tfrac{1}{2}K(p_a,p_b).$$

The details for the evaluation of this integral are given in Appendix D. The results are as follows:

$$I(p,p)=1+\ln(m/2\bar{\omega})\,, \tag{III.18}$$

$$I(q,q)=\ln(E/\bar{\omega})\,, \tag{III.19}$$

$$I(p',p')=\ln(\mathcal{E}'/\bar{\omega})\,, \tag{III.20}$$

$$I(q',q')=\ln[(E-\mathcal{E}')/\bar{\omega}]\,, \tag{III.21}$$

$$I(p,p')=\ln(\mathcal{E}'/\bar{\omega})\ln(2\mathcal{E}'/m)-\tfrac{1}{4}\ln^2(m/2\mathcal{E}')\,, \tag{III.22}$$

$$I(p,q)=\ln\frac{E}{\bar{\omega}}\ln\frac{2E}{\mu}-\tfrac{1}{4}\ln^2\frac{m}{2E}+\tfrac{1}{2}\Phi(1)-\tfrac{1}{2}\Phi\left(1-\frac{\mu^2}{2mE}\right), \tag{III.23}$$

$$I(p,q')=\ln\frac{E-\mathcal{E}'}{\bar{\omega}}\ln\frac{2(E-\mathcal{E}')}{\mu}-\tfrac{1}{4}\ln^2\frac{m}{2(E-\mathcal{E}')}+\tfrac{1}{2}\Phi(1)-\tfrac{1}{2}\Phi\left(1-\frac{\mu^2}{2m(E-\mathcal{E}')}\right), \tag{III.24}$$

$$I(p',q)=\ln\frac{\mathcal{E}'}{\bar{\omega}}\ln\frac{2(E-\mathcal{E}')}{\mu}-\tfrac{1}{2}\Phi\left(\frac{\mathcal{E}'}{E}\right)+\tfrac{1}{2}\Phi(1)-\tfrac{1}{4}\ln^2\frac{\mathcal{E}'}{E}+\tfrac{1}{2}\ln\frac{E}{\mathcal{E}'}\ln\frac{E-\mathcal{E}'}{\mathcal{E}'}$$
$$-\tfrac{1}{2}\Phi\left(\frac{T_0}{4m\mu^2(E-\mathcal{E}')}\right)+\tfrac{1}{2}\Phi\left(\frac{T_0}{8m^2(E-\mathcal{E}')^2}\right)+\tfrac{1}{4}\ln\left(\frac{2m(E-\mathcal{E}')}{\mu^2}\right)\ln\left(\frac{2m(E-\mathcal{E}')^3\mu^2}{\mathcal{E}'^2(\mu^2-2m(E-\mathcal{E}'))^2}\right), \tag{III.25}$$

$$I(p',q')=\ln\frac{E}{\bar{\omega}}\ln\frac{2E}{\mu}-\tfrac{1}{2}\Phi\left(\frac{\mathcal{E}'}{E-\mathcal{E}'}\right)+\tfrac{1}{2}\Phi(1)+\tfrac{1}{2}\ln\frac{E-\mathcal{E}'}{\mathcal{E}'}\ln\frac{E-2\mathcal{E}'}{\mathcal{E}'}-\tfrac{1}{4}\ln^2\left(\frac{E-\mathcal{E}'}{\mathcal{E}'}\right)$$
$$-\tfrac{1}{2}\Phi\left(\frac{8m^2E(E-2\mathcal{E}')}{T_0}\right)+\tfrac{1}{2}\Phi\left(\frac{4m\mu^2(E-2\mathcal{E}')}{T_0}\right)-\tfrac{1}{2}\ln\left(\frac{2mE}{\mu^2}\right)\ln\left(\frac{T_0}{4m\mathcal{E}'(2mE-\mu^2)}\right),\quad \text{(III.26)}$$

$$I(q,q')=\frac{\kappa'+1}{2a'}\left\{2\ln\left(\frac{E-\mathcal{E}'}{\bar{\omega}}\right)\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right)-\tfrac{1}{2}\ln^2\left(\frac{E}{E-\mathcal{E}'}\right)+\ln\left(\frac{E+\mathcal{E}'r_1}{E-\mathcal{E}'}\right)\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right)\right.$$
$$\left.+\ln\left(\frac{E}{E-\mathcal{E}'}\right)\ln\left(\frac{\mathcal{E}'(1+r_1)}{E-\mathcal{E}'}\right)+\Phi\left(1-\frac{r_1\mathcal{E}'}{E-\mathcal{E}'}\right)-\Phi\left(1-\frac{(1+r_1)\mathcal{E}'}{E}\right)+\Phi\left(\frac{r_1\mathcal{E}'}{E+\mathcal{E}'r_1}\right)-\Phi\left(\frac{(1+r_1)\mathcal{E}'}{E+\mathcal{E}'r_1}\right)\right\},\quad \text{(III.27)}$$

where $r_1=(a'-\kappa')/2\kappa'$. It has consistently been our policy to reduce the arguments of all Spence functions so that their absolute value be less than unity. Equations (A52)–(A55) have been used to accomplish this.

B. Hard Photons

Before we can proceed to perform the integrations of Eq. (III.14) for the hard photon cross section we must first do the trace calculation of Eq. (III.5) for A. This is quite complicated but nonetheless straightforward. We list the result of this calculation as follows:

$$A=\sum_{i=1}^{3}\sum_{\alpha}A_{\alpha}{}^{(i)}\quad \alpha\epsilon\{1,p,p',q,q'\}^2,\quad \text{(III.28)}$$

where

$$A_{q^2}{}^{(1)}=\left(\frac{1}{p_3{}^2}\right)^2\frac{\mu^2}{(q\cdot k)^2}\times[-T_0+4pk(p_3{}^2+4pq-2pk)],\quad \text{(III.29)}$$

$$A_{q'^2}{}^{(1)}=\left(\frac{1}{p_3{}^2}\right)^2\frac{\mu^2}{(q'k)^2}[-T_0],\quad \text{(III.30)}$$

$$A_{qq'}{}^{(1)}=\left(\frac{1}{p_3{}^2}\right)^2\frac{p_3{}^2+2\mu^2}{(q\cdot k)(q'\cdot k)}\times[T_0-2pk(p_3{}^2+4pq)],\quad \text{(III.31)}$$

$$A_{q}{}^{(1)}=\left(\frac{1}{p_3{}^2}\right)^2\frac{4}{qk}[p_3{}^2(\mu^2-pq+pk)],\quad \text{(III.32)}$$

$$A_{q'}{}^{(1)}=\left(\frac{1}{p_3{}^2}\right)^2\frac{2}{q'\cdot k}[-p_3{}^2(p_3{}^2+2pq+2\mu^2)],\quad \text{(III.33)}$$

$$A_1{}^{(1)}=(1/p_3{}^2)^2 4[p_3{}^2-2m^2],\quad \text{(III.34)}$$

$$A_{p^2}{}^{(2)}=\left(\frac{1}{q_3{}^2}\right)^2\frac{m^2}{(pk)^2}[-T_0+4qk(p_3{}^2+4pq-2qk)+4p'k(\mu^2-2pq+2qk)],\quad \text{(III.35)}$$

$$A_{p'^2}{}^{(2)}=\left(\frac{1}{q_3{}^2}\right)^2\frac{m^2}{(p'k)^2}[-T_0+4pk(2pq-\mu^2)],\quad \text{(III.36)}$$

$$A_{pp'}{}^{(2)}=\left(\frac{1}{q_3{}^2}\right)^2\frac{2}{(pk)(p'k)}\times[(pp')[-T_0+2qk(p_3{}^2+4pq-2qk)]-4m^2(qk)^2],\quad \text{(III.37)}$$

$$A_{p}{}^{(2)}=\left(\frac{1}{q_3{}^2}\right)^2\frac{2}{pk}[T_0+2p_3{}^2(pq)+6m^2(2pq-\mu^2)-4qk(p_3{}^2+2m^2+2pq-qk)-2p'k(\mu^2-2pq+2qk)],\quad \text{(III.38)}$$

$$A_{p'}{}^{(2)}=\left(\frac{1}{q_3{}^2}\right)^2\frac{2}{p'\cdot k}\times[-T_0-p_3{}^2(p_3{}^2+6pq)+6m^2(\mu^2-2pq)+2pk(2p_3{}^2-\mu^2+6pq-2pk-4qk)+2qk(3p_3{}^2+4pq-2qk+2m^2)],\quad \text{(III.39)}$$

$$A_1{}^{(2)}=(1/q_3{}^2)^2\times 8[-p_3{}^2-4pq+2pk-p'k+3qk],\quad \text{(III.40)}$$

$$A_{pq}{}^{(3)}=\frac{1}{q_3{}^2p_3{}^2}\frac{2pq}{(pk)(qk)}[T_0-2p'k(\mu^2-2pq)],\quad \text{(III.41)}$$

$$A_{p'q'}{}^{(3)}=\frac{1}{q_3{}^2p_3{}^2}\frac{2pq}{(p'k)(q'k)}[T_0-2pk(2pq-\mu^2)],\quad \text{(III.42)}$$

$$A_{pq'}{}^{(3)}=\frac{1}{q_3{}^2p_3{}^2}\frac{(2pq+p_3{}^2)}{(pk)(q'k)}\times[-T_0+2qk(p_3{}^2+\mu^2+2pq)],\quad \text{(III.43)}$$

$$A_{p'q}{}^{(3)}=\frac{1}{q_3{}^2p_3{}^2}\frac{(2pq+p_3{}^2-2pk)}{(p'k)(qk)}\times[T_0+2pk(p_3{}^2+6pq-\mu^2-2pk)],\quad \text{(III.44)}$$

$$A_q^{(3)}=\frac{1}{q_3^2p_3^2}\frac{(-2)}{(qk)}[T_0+(4pq+\mu^2)(4pq+p_3^2)-4pk(p_3^2+\mu^2+6pq-2pk)+2p'k(4pq-2pk-\mu^2)], \quad \text{(III.45)}$$

$$A_{q'}^{(3)}=\frac{1}{q_3^2p_3^2}\frac{2}{(q'k)}[(p_3^2+2pq+2\mu^2)\times(p_3^2+2pq)+2pq\mu^2], \quad \text{(III.46)}$$

$$A_p^{(3)}=\frac{1}{q_3^2p_3^2}\frac{(-2)}{(pk)}\times[p_3^2(\mu^2+2m^2)+2pq(\mu^2+2pq+4m^2)+2qk(p_3^2-4m^2)+4p'k(pq+m^2)], \quad \text{(III.47)}$$

$$A_{p'}^{(3)}=\frac{1}{q_3^2p_3^2}\frac{2}{(p'k)}[p_3^2(p_3^2+4pq+2m^2)+8m^2(pq)-2pk(3p_3^2+6pq+2m^2-4pk)-2qk(p_3^2-2pk)], \quad \text{(III.48)}$$

$$A_1^{(3)}=\frac{4}{q_3^2p_3^2}[p_3^2+8pq+4m^2-6pk-2qk+2p'k], \quad \text{(III.49)}$$

where $q_3=p_3-k=q'-q$.

As in the soft photon case we now write

$$d\sigma^h=(\alpha/\pi)d\sigma_0\delta_h. \quad \text{(III.50)}$$

It then follows from comparing Eq. (III.14) with Eq. (II.9) that

$$\delta_h=\sum_{i=1}^{3}\sum_{\alpha}\mathcal{G}_\alpha{}^{(i)}\quad \alpha\epsilon\{1,p,p',q,q'\}^2, \quad \text{(III.51)}$$

where

$$\mathcal{G}_\alpha{}^{(i)}=\frac{(p_3^2)^2}{4\pi T_0}\int_{\bar\omega}^{\Delta E}\omega d\omega\int_{E-|\mathbf{q}|}^{\bar y}\frac{dy}{|\mathbf{q}-\mathbf{k}|}\int_0^{2\pi}A_\alpha{}^{(i)}d\varphi.$$

These integrations are extremely complicated and require much care. They are evaluated in Appendix E where the results are accurate only up to terms of order unity. If we write

$$d\sigma_{\text{inel}}=(\alpha/\pi)d\sigma_0\delta_{\text{inel}}, \quad \text{(III.52)}$$

then from Eqs. (III.51) and (III.17)

$$\delta_{\text{inel}}+\delta_\lambda=\sum_{a\le b}J(p_a,p_b)+\sum_a J(p_a)\equiv\delta_{\text{inel}}', \quad \text{(III.53)}$$

where

$$J(p_a,p_b)=\sum_{i=1}^{3}\mathcal{G}_{p_ap_b}{}^{(i)}+\epsilon_{ab}I(p_a,p_b),\qquad J(p_a)=\sum_{i=1}^{3}\mathcal{G}_{p_a}{}^{(i)}. \quad \text{(III.54)}$$

That is, we have combined corresponding terms in the hard and soft photon cross sections. We now list these combined terms:

$$J(p,p)=\ln(m/2\bar\omega), \quad \text{(III.55)}$$

$$J(q,q)=\ln(E/\Delta \mathrm{E}), \quad \text{(III.56)}$$

$$J(p',p')=\ln(\mathcal{E}'/\Delta E), \quad \text{(III.57)}$$

$$J(q',q')=\ln[(E-\mathcal{E}')/\Delta E], \quad \text{(III.58)}$$

$$J(p,p')=2\ln\frac{\Delta E}{\mathcal{E}'}\ln\frac{2\mathcal{E}'}{m}+\tfrac12\ln^2\left(\frac{2\mathcal{E}'}{m}\right)-\tfrac12\ln^2\left(\frac{\Delta E}{\bar\omega}\right)+R\ln\xi-\Phi\left(\frac{-\Delta E}{\mathcal{E}'}\right)+\left[\frac{(p_3^2)^2}{T_0}-1\right]I_1-\left[\frac{(p_3^2)^2}{T_0}+\tfrac12\right]I_2-\frac{4m^2E(E-3\mathcal{E}')}{3T_0}I_3-\frac{m^2E^2}{T_0}I_4+\left[\frac{T_0-4m^2E(E-\mathcal{E}')}{4m\mu^2\mathcal{E}'}+\left(\frac{4m^2E^2}{T_0}-1\right)\ln\rho\right]\ln\frac{\Delta E}{\bar\omega}, \quad \text{(III.59)}$$

$$J(p,q)=2\ln\frac{E}{\Delta E}\ln\frac{2E}{\mu}-\tfrac12\ln^2\left(\frac{2E}{m}\right)+\tfrac12\ln^2\frac{\Delta E}{\bar\omega}+\Phi(1)+\left(1+\frac{p_3^2(\mu^2+2mE)}{T_0}\right)\ln\rho\ln\frac{\Delta E}{\bar\omega}-\Phi[1-(\mu^2/2mE)], \quad \text{(III.60)}$$

$$J(p,q')=2\ln\left(\frac{\Delta E}{E-\mathcal{E}'}\right)\ln\left(\frac{2(E-\mathcal{E}')}{\mu}\right)+\tfrac12\ln^2\left(\frac{2(E-\mathcal{E}')}{m}\right)-\tfrac12\ln^2\frac{\Delta E}{\bar\omega}-\left[1+\frac{2mE(p_3^2+\mu^2-2mE)}{T_0}\right]\ln\rho\ln\frac{\Delta E}{\bar\omega}-\Phi(1)+\Phi\left(1-\frac{\mu^2}{2m(E-\mathcal{E}')}\right), \quad \text{(III.61)}$$

$$J(p',q)=2\ln\frac{\Delta E}{\mathcal{E}'}\ln\frac{2(E-\mathcal{E}')}{\mu}+2\left[1+\frac{p_3{}^2(\mu^2+6mE)}{T_0}\right]\ln\xi\ln\frac{2E}{\mu}+\Phi\left(\frac{\mathcal{E}'}{E}\right)-\Phi(1)+\tfrac{1}{2}\ln^2\left(\frac{E}{\mathcal{E}'}\right)-\ln\left(\frac{E}{\mathcal{E}'}\right)\ln\left(\frac{E-\mathcal{E}'}{\mathcal{E}'}\right)$$
$$+\Phi\left(\frac{T_0}{4m\mu^2(E-\mathcal{E}')}\right)-\Phi\left(\frac{T_0}{8m^2(E-\mathcal{E}')^2}\right)-\tfrac{1}{2}\ln\frac{2m(E-\mathcal{E}')}{\mu^2}\ln\left(\frac{2m(E-\mathcal{E}')^3\mu^2}{\mathcal{E}'^2[\mu^2-2m(E-\mathcal{E}')]^2}\right), \quad \text{(III.62)}$$

$$J(p',q')=2\ln\frac{\mathcal{E}'}{\Delta E}\ln\frac{2E}{\mu}-2\left[1+\frac{p_3{}^2(\mu^2+2mE)}{T_0}\right]\ln\xi\ln\frac{2E}{\mu}+\Phi(1)-\Phi\left(\frac{\mathcal{E}'}{E-\mathcal{E}'}\right)-\tfrac{1}{2}\ln^2\frac{E-\mathcal{E}'}{\mathcal{E}'}+\ln\frac{E-\mathcal{E}'}{\mathcal{E}'}\ln\frac{E-2\mathcal{E}'}{\mathcal{E}'}$$
$$+\Phi\left(\frac{4m\mu^2(E-2\mathcal{E}')}{T_0}\right)-\Phi\left(\frac{8m^2E(E-2\mathcal{E}')}{T_0}\right)+\ln\left(\frac{2mE}{\mu^2}\right)\ln\left|\frac{T_0}{4m\mathcal{E}'(2mE-\mu^2)}\right|, \quad \text{(III.63)}$$

$$J(q,q')=\frac{-(\kappa'+1)}{a'}\left\{2\ln\left(\frac{E-\mathcal{E}'}{\Delta E}\right)\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right)+\ln\left(\frac{E+\mathcal{E}'r_1}{E-\mathcal{E}'}\right)\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right)+\ln\left(\frac{E}{E-\mathcal{E}'}\right)\ln\left(\frac{\mathcal{E}'(1+r_1)}{E-\mathcal{E}'}\right)\right.$$
$$\left.-\tfrac{1}{2}\ln^2\frac{E}{E-\mathcal{E}'}+\Phi\left(1-\frac{r_1\mathcal{E}'}{E-\mathcal{E}'}\right)-\Phi\left(1-\frac{(1+r_1)\mathcal{E}'}{E}\right)+\Phi\left(\frac{r_1\mathcal{E}'}{E+\mathcal{E}'r_1}\right)-\Phi\left(\frac{(1+r_1)\mathcal{E}'}{E+\mathcal{E}'r_1}\right)\right\}, \quad \text{(III.64)}$$

$$J(p)=-\left(\ln\frac{\Delta E}{\bar{\omega}}\right)\left[\frac{mE(E-\mathcal{E}')}{\mu^2\mathcal{E}'}+(\ln\rho)\left[\frac{2p_3{}^2(\mu^2+mE)+2mE(2mE-\mu^2)}{T_0}\right]\right], \quad \text{(III.65)}$$

$$J(p')=\frac{-p_3{}^2(\mu^2+2mE+2m\mathcal{E}')}{T_0}I_1+\left[\frac{(p_3{}^2)^2}{T_0}+\tfrac{1}{4}\right]I_2-\frac{4m^2E\mathcal{E}'}{T_0}I_3$$
$$+\frac{m^2E^2}{T_0}I_4-\frac{4mEp_3{}^2}{T_0}\left[R\ln\xi+\Phi\left(\frac{-\Delta E}{\mathcal{E}'}\right)\right]-\frac{(p_3{}^2)^2}{T_0}\frac{\Delta E}{\mathcal{E}'}(R-1-\ln\xi), \quad \text{(III.66)}$$

$$\xi=\mathcal{E}'/(\mathcal{E}'+\Delta E),\quad R=2\ln(2\mathcal{E}'/m)-\ln(\Delta E/\bar{\omega})-\ln\xi,\ \rho=\mu^2\mathcal{E}'/2mE(E-\mathcal{E}'),$$
$$I_1=(1-\xi)(R+2)+\xi\ln\xi,\qquad I_3=(1-\xi^3)(R+\tfrac{2}{3})+\xi^3\ln\xi+(1-\xi)+\tfrac{1}{2}(1-\xi^2),$$
$$I_2=(1-\xi^2)(R+1)+\xi^2\ln\xi+1-\xi,\quad I_4=(1-\xi^4)(R+\tfrac{1}{2})+\xi^4\ln\xi+(1-\xi)+\tfrac{1}{2}(1-\xi^2)+\tfrac{1}{3}(1-\xi^3),$$

and all other quantities have already been defined. Finally, upon performing the required summation in Eq. (III.53), we get

$$\delta_{\text{inel}}'=\tfrac{1}{2}\ln^2\frac{2\mathcal{E}'}{m}-\tfrac{1}{2}\ln^2\frac{\Delta E}{\bar{\omega}}-2\ln\frac{\mathcal{E}'}{\Delta E}\ln\frac{2\mathcal{E}'}{m}+\ln\left(\frac{mE}{2\bar{\omega}\Delta E}\right)+\ln\frac{(E-\mathcal{E}')\mathcal{E}'}{(\Delta E)^2}\left[1+2\ln\left(\frac{E}{E-\mathcal{E}'}\right)\right]-\frac{\kappa'+1}{a'}\ln\frac{E(E-\mathcal{E}')}{(\Delta E)^2}\ln\frac{a'+\kappa'}{a'-\kappa'}$$
$$-\ln\left(\frac{E}{E-\mathcal{E}'}\right)\left[\ln\frac{2\mathcal{E}'}{m}-2\ln\frac{2E}{\mu}\right]-[1+\tfrac{1}{2}\ln\rho]\ln\frac{\Delta E}{\bar{\omega}}+R\ln\xi-\left[1+\frac{p_3{}^2(\mu^2+2mE)}{T_0}\right]I_1-\tfrac{1}{4}I_2-\frac{4m^2E^2}{3T_0}I_3$$
$$+\left\{\frac{4p_3{}^2mE}{T_0}\ln\xi\left[2\ln\frac{2E}{\mu}-R\right]-\frac{(p_3{}^2)^2}{T_0}\frac{\Delta E}{\mathcal{E}'}(R-1-\ln\xi)+\ln\frac{E-\mathcal{E}'}{\mathcal{E}'}\ln\frac{E-2\mathcal{E}'}{E-\mathcal{E}'}-\tfrac{1}{2}\ln^2\frac{2m(E-\mathcal{E}')}{\mu^2}\right.$$
$$+\ln\frac{2m(E-\mathcal{E}')}{\mu^2}\ln\left(1-\frac{2m(E-\mathcal{E}')}{\mu^2}\right)+\ln\frac{2mE}{\mu^2}\ln\left[\frac{(E-\mathcal{E}')^{-1}T_0}{4m(2mE-\mu^2)}\right]-\frac{\kappa'+1}{a'}\left[\ln\frac{E+\mathcal{E}'r_1}{E}\ln\frac{a'+\kappa'}{a'-\kappa'}\right.$$
$$\left.+\ln\frac{E}{E-\mathcal{E}'}\ln\frac{\mathcal{E}'(1+r_1)}{E-\mathcal{E}'}-\tfrac{1}{2}\ln^2\frac{E}{E-\mathcal{E}'}+\Phi\left(1-\frac{r_1\mathcal{E}'}{E-\mathcal{E}'}\right)-\Phi\left(1-\frac{(1+r_1)\mathcal{E}'}{E}\right)+\Phi\left(\frac{r_1\mathcal{E}'}{E+\mathcal{E}'r_1}\right)-\Phi\left(\frac{(1+r_1)\mathcal{E}'}{E+\mathcal{E}'r_1}\right)\right]$$
$$+\Phi\left(1-\frac{\mu^2}{2m(E-\mathcal{E}')}\right)-\Phi\left(1-\frac{\mu^2}{2mE}\right)+\Phi\left(\frac{T_0}{4m\mu^2(E-\mathcal{E}')}\right)-\Phi\left(\frac{T_0}{8m^2(E-\mathcal{E}')^2}\right)+\Phi\left(\frac{\mathcal{E}'}{E}\right)$$
$$\left.-\Phi\left(\frac{\mathcal{E}'}{E-\mathcal{E}'}\right)+\Phi\left(\frac{4m\mu^2(E-2\mathcal{E}')}{T_0}\right)-\Phi\left(\frac{8m^2E(E-2\mathcal{E}')}{T_0}\right)-\left[1+\frac{4mEp_3{}^2}{T_0}\right]\Phi\left(-\frac{\Delta E}{\mathcal{E}'}\right)\right\}. \quad \text{(III.67)}$$

The latter group of terms bracketed by { } are all of order unity. Moreover, their sum is also of order unity and can safely be neglected. To be on the safe side we shall keep them and in column 2 of Table I we list some values of { }.

IV. RESULTS

The observable scattering cross section $d\sigma$ is given by $d\sigma = d\sigma_{\rm el} + d\sigma_{\rm inel}$. From Eqs. (II.26) and (III.52) we therefore get

$$\frac{d\sigma}{d\mathcal{E}'} = \frac{d\sigma_0}{d\mathcal{E}'}\left(1+\frac{\alpha}{\pi}\delta\right), \tag{IV.1}$$

where $\delta = \delta_1 + \delta_2 + \delta_{\rm inel}'$. [$(\alpha/\pi)\delta$ is called the radiative correction] and δ_1, δ_2, $\delta_{\rm inel}'$ are given in Eqs. (II.30), (II.31), and (III.67), respectively.

It can be seen from Eq. (III.67) that δ contains a term $l_1 = \frac{1}{2}\ln^2(2\mathcal{E}'/m)$. As we have already indicated in the introduction, this worrisome term should always be grouped with the second term of Eq. (III.29), viz., $l_2 = -\frac{1}{2}\ln^2(\Delta E/\bar{\omega})$. By this we mean the following: In attempting to trace the origin of l_1 we find that it comes from $J(p,p')$ [Eq. (III.59)]. We further notice that $J(p,q)$ and $J(p,q')$ [Eqs. (III.60) and (III.61)] also contain terms like l_1, but in summing, these terms have cancelled. Moreover, all three equations contain l_2, and we see that terms like l_1 always appear together with l_2. To understand this better, consider the y itegrations of cases 2(c), 3(a), and 3(c) of Appendix E. A moment's thought shows that the l_2 terms come directly from using $\bar{y}$ for the upper limit of these integrations. Had we used $2E$ instead of $\bar{y}$, i.e., done an isotropic integration thereby ignoring the kinematic restriction, the l_2 term would be absent. Hence, terms like l_1 can be viewed as arising from estimating the allowed photon region by the isotropic, experimentally restricted region, and l_2 as the correction to this overestimate when the kinematic conditions are taken into account.

Summarizing, we can say that the l_1 term alone would be present in a radiative correction, not so much because of the improper handling of the experimental conditions, but rather because of the inaccurate accounting of the kinematic conditions, whenever such a distinction is possible.

We have used the technique of infrared extraction developed by Yennie *et al.*[13] in evaluating the two-photon exchange diagrams. It is generally assumed that after having made this extraction, what remains (δ_J) contributes negligibly to δ. It is also generally assumed that δ_K defined by Eq. (II.24) is likewise of order unity. We have evaluated both δ_J, Eq. (C11) and δ_K, Eq. (A45), and have found them to be substantially larger than unity. However, $\delta_2 = \delta_J + \delta_K$ is of order unity.

In Table I we give some numerical results for the case $E = 20$ BeV, $\Delta E = (E - \mathcal{E}')/20$.

The radiative corrections $(\alpha/\pi)\hat{\delta}$, for π^+-e scattering can easily be derived from our present results. By counting the number of single-meson corners per Feynman diagram in Fig. 1 we see that only the sign of the contribution from the two-photon exchange diagrams [(2), (3), and (4)] gets changed. Similar considerations show that for the inelastic case only the signs of the cross terms get changed. Hence, we can write

$$\hat{\delta} = \delta_1 - \delta_2 + \hat{\delta}_{\rm inel}', \tag{IV.2}$$

where

$$\begin{aligned}
\hat{\delta}_{\rm inel}' = {} & \tfrac{1}{2}\ln^2\frac{2\mathcal{E}'}{m} - \tfrac{1}{2}\ln^2\frac{\Delta E}{\bar{\omega}} - 2\ln\frac{\mathcal{E}'}{\Delta E}\ln\frac{2\mathcal{E}'}{m} + \ln\frac{mE}{2\bar{\omega}\Delta E} + \ln\frac{(E-\mathcal{E}')\mathcal{E}'}{(\Delta E)^2}\left[1 - 2\ln\frac{E}{E-\mathcal{E}'}\right] + \ln\frac{E}{E-\mathcal{E}'}\left[\ln\frac{2\mathcal{E}'}{m} - 2\ln\frac{2E}{\mu}\right] \\
& - \frac{\kappa'+1}{a'}\ln\frac{E(E-\mathcal{E}')}{(\Delta E)^2}\ln\frac{a'+\kappa'}{a'-\kappa'} + R\ln\xi - \left[1+\tfrac{1}{2}\ln\rho\right]\ln\frac{\Delta E}{\bar{\omega}} - \left(1 + \frac{p_3^2(\mu^2+2mE)}{T_0}\right)I_1 - \tfrac{1}{4}I_2 - \frac{4m^2E^2}{3T_0}I_4 \\
& - \left\{\frac{4p_3^2mE}{T_0}\ln\xi\left[2\ln\frac{2E}{\mu} - R\right] - \frac{(p_3^2)^2}{T_0}\frac{\Delta E}{\mathcal{E}'}(R - 1 - \ln\xi) + \ln\frac{E-\mathcal{E}'}{\mathcal{E}'}\ln\frac{E-2\mathcal{E}'}{E-\mathcal{E}'} - \tfrac{1}{2}\ln^2\frac{2m(E-\mathcal{E}')}{\mu^2}\right. \\
& + \ln\frac{2m(E-\mathcal{E}')}{\mu^2}\ln\left[1 - \frac{2m(E-\mathcal{E}')}{\mu^2}\right] + \ln\frac{2mE}{\mu^2}\ln\left(\frac{T_0}{4m(2mE-\mu^2)(E-\mathcal{E}')}\right) + \frac{\kappa'+1}{a'}\left[\ln\frac{E+\mathcal{E}'r_1}{E}\ln\frac{a'+\kappa'}{a'-\kappa'}\right. \\
& \left. + \ln\frac{E}{E-\mathcal{E}'}\ln\frac{\mathcal{E}'(1+r_1)}{E-\mathcal{E}'} - \tfrac{1}{2}\ln^2\frac{E}{E-\mathcal{E}'} + \Phi\left(1 - \frac{r_1\mathcal{E}'}{E-\mathcal{E}'}\right) - \Phi\left(1 - \frac{(1+r_1)\mathcal{E}'}{E}\right) + \Phi\left(\frac{r_1\mathcal{E}'}{E+\mathcal{E}'r_1}\right) - \Phi\left(\frac{(1+r_1)\mathcal{E}'}{E+\mathcal{E}'r_1}\right)\right] \\
& + \Phi\left(1 - \frac{\mu^2}{2m(E-\mathcal{E}')}\right) - \Phi\left(1 - \frac{\mu^2}{2mE}\right) + \Phi\left(\frac{T_0}{4m\mu^2(E-\mathcal{E}')}\right) - \Phi\left(\frac{T_0}{8m^2(E-\mathcal{E}')^2}\right) + \Phi\left(\frac{\mathcal{E}'}{E}\right) - \Phi\left(\frac{\mathcal{E}'}{E-\mathcal{E}'}\right) \\
& \left. + \Phi\left(\frac{4m\mu^2(E-2\mathcal{E}')}{T_0}\right) - \Phi\left(\frac{8m^2E(E-2\mathcal{E}')}{T_0}\right) + \left[1 - \frac{4mEp_3^2}{T_0}\right]\Phi\left(-\frac{\Delta E}{\mathcal{E}'}\right)\right\}.
\end{aligned} \tag{IV.3}$$

TABLE I. Some numerical results for $E=20$ BeV and $\Delta E=(E-\mathcal{E}')/20$.

1 $\mathcal{E}'(10^4$ MeV)	2 { }[a]	3 δ_1[b]	4 δ_2[b]	5 δ_{inel}'[b]	6 $(\alpha/\pi)\delta(\%)$[c]	7 $\hat{\delta}_{inel}'$[d]	8 $(\alpha/\pi)\hat{\delta}(\%)$[e]
0.1	0.477	14.847	−2.110	−10.617	0.49	−8.470	1.97
0.2	−0.075	16.383	−2.787	−16.291	−0.63	−15.642	0.82
0.3	−0.383	17.295	−3.229	−21.046	−1.62	−21.560	−0.24
0.4	−0.614	17.951	−3.547	−25.253	−2.52	−27.086	−1.30
0.5	−0.816	18.467	−3.783	−29.303	−3.39	−32.561	−2.39
0.6	−1.009	18.894	−3.959	−33.439	−4.29	−38.250	−3.58
0.7	−1.195	19.260	−4.085	−37.971	−5.29	−44.491	−4.91
0.8	−1.419	19.581	−4.157	−43.557	−6.53	−51.921	−6.55
0.9	−1.732	19.867	−4.193	−52.309	−8.52	−62.478	−8.92

[a] In this column we have listed the bracketed terms of Eq. (III.67).
[b] δ_1, δ_2 and δ_{inel}' (columns 3, 4, 5) are given by Eqs. (II.30), (II.31), and (III.67), respectively.
[c] $(\alpha/\pi)\delta=(\alpha/\pi)(\delta_1+\delta_2+\delta_{inel}')$ is the radiative correction for π^--e scattering.
[d] $\hat{\delta}_{inel}'$ is given by Eq. (IV.3).
[e] $(\alpha/\pi)\hat{\delta}=(\alpha/\pi)(\hat{\delta}_1-\delta_2+\delta_{inel}')$ is the radiative correction for π^+-e scattering.

We have listed some numerical results for $\hat{\delta}_{inel}'$ and $\hat{\delta}$ in the last two columns of Table I.[16]

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APPENDIX A: INTEGRATIONS

In this section we outline the techniques for performing the integrations occurring in the elastic cross section. The methods employed are essentially the same as those used by Brown and Feynman.[17] Those integrals which suffer an ultraviolet divergence are handled by appending to the integrand the regulator $\Lambda^2/(k^2+\Lambda^2)$ which is ultimately to be considered in the limit of $\Lambda\to\infty$. And, as has already been indicated, a small fictitious mass λ is assigned to the photon in order to take care of the infrared divergence. We ultimately take the limit $\lambda\to 0$.

All the integrals in this section have denominators which are a product of various factors of the form $(k^2-2p_a\cdot k+\Delta_a)$. We can combine such factors by making use of the relations[18]

$$\frac{1}{AB}=\int_0^1\frac{dy}{[Ay+B(1-y)]^2}, \quad \text{(A1)}$$

$$\frac{1}{A^2B}=\int_0^1\frac{2ydy}{[Ay+B(1-y)]^3}, \quad \text{(A2)}$$

$$\frac{1}{A^3B}=\int_0^1\frac{3y^2dy}{[Ay+B(1-y)]^4}. \quad \text{(A3)}$$

In this manner we can reduce all the integrals in this section to a form where the d^4k part of the integration is simply given by one of the following:

$$\frac{8i}{(2\pi)^2}\int\frac{d^4k(1;k_\nu)}{[k^2-2pk+\Delta]^3}=\frac{(1;p_\nu)}{p^2-\Delta}, \quad \text{(A4)}$$

$$\frac{24i}{(2\pi)^2}\int\frac{d^4k(1;k_\nu;k_\nu k_\mu)}{[k^2-2pk+\Delta]^4} =\frac{-(1;p_\nu;[p_\nu p_\mu-g_{\mu\nu}(p^2-\Delta)/2])}{(p^2-\Delta)^2}. \quad \text{(A5)}$$

A. Two-Denominator Integrals

The following is a complete list of the two-denominator integrals arising in our calculation:

$$L_{(0;\nu)}{}^{(\mu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2qk)(k^2-2q'k)},$$

$$L_{(0;\nu)}{}^{(m)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2pk)(k^2-2p'k)},$$

$$P=\frac{i}{(2\pi)^2}\int\frac{d^4k}{k^2(k^2-2pk)},$$

$$P'=\frac{i}{(2\pi)^2}\int\frac{d^4k}{(k-p_3)^2(k^2-2pk)},$$

$$R=\frac{i}{(2\pi)^2}\int\frac{d^4k}{k^2(k-p_3)^2}.$$

The method for evaluating these integrals is not very different than the method described in Ref. 17, Appendix Y(a). We therefore simply state the results.

$$L_0{}^{(m)}=\tfrac{1}{4}[\ln(m^2/\Lambda^2)-1+\tfrac{1}{2}(p+p')^2\mu(p,p')], \quad \text{(A6)}$$

$$L_0{}^{(\mu)}=\tfrac{1}{4}[\ln(\mu^2/\Lambda^2)-1+\tfrac{1}{2}Q^2\mu(q,q')], \quad \text{(A7)}$$

[16] P. B. Allen and M. M. Sternheim, Brookhaven Internal Report, BNL 7588, 1963 (unpublished), have checked numerically the accuracy of some of the approximations employed in Appendix E.
[17] L. M. Brown and R. P. Feynman, Phys. Rev. 85, 231 (1952).
[18] R. P. Feynman, Phys. Rev. 76, 769 (1949).

$$L_\nu^{(m)}=\tfrac{1}{8}(p+p')_\nu[\ln(m^2/\Lambda^2)-\tfrac{1}{2}+\tfrac{1}{2}(p+p')^2\mu(p,p')], \quad (A8)$$

$$L_\nu^{(\mu)}=\tfrac{1}{8}Q_\nu[\ln(\mu^2/\Lambda^2)-\tfrac{1}{2}+\tfrac{1}{2}Q^2\mu(q,q')], \quad (A9)$$

$$P=P'=\tfrac{1}{4}[\ln(m^2/\Lambda^2)-1], \quad (A10)$$

$$R=\tfrac{1}{4}[\ln(p_3^2/\Lambda^2)-1]. \quad (A11)$$

B. Three-Denominator Integrals

We shall need to know the following three-denominator integrals:

$$p_3^\nu S_\nu^{(\mu)}=\frac{i}{(2\pi)^2}\int\frac{(p_3k)d^4k}{(k^2-2qk)^2(k^2-2q'k)},$$

$$p_3^\nu S_\nu^{(m)}=\frac{i}{(2\pi)^2}\int\frac{(p_3k)d^4k}{(k^2-2pk)^2(k^2-2p'k)},$$

$$K_{(0;\nu)}^{(\mu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2qk)(k^2-2q'k)(k^2+\lambda^2)},$$

$$K_{(0;\nu;\nu\sigma)}^{(m)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu;k_\nu k_\sigma)d^4k}{(k^2-2pk)(k^2-2p'k)(k^2+\lambda^2)},$$

$$G_{(0;\nu)}^{(\mu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{k^2(k-p_3)^2(k^2-2q'\cdot k)},$$

$$G_{(0;\nu)}^{m}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{k^2(k-p_3)^2(k^2-2pk)}.$$

Again we simply list the results.

$$p_3^\nu S_\nu^{(\mu)}=\tfrac{1}{4}[\tfrac{1}{2}Q^2\mu(q,q')-2], \quad (A12)$$

$$p_3^\nu S_\nu^{(m)}=\tfrac{1}{4}[2-\tfrac{1}{2}(p+p')^2\mu(p,p')], \quad (A13)$$

$$K_0^{(\mu)}=K(q,q')/8(q\cdot q'), \quad (A14)$$

$$K_0^{(m)}=K(p,p')/8(p\cdot p'), \quad (A15)$$

$$K_\nu^{(\mu)}=\tfrac{1}{8}Q_\nu\mu(q,q'), \quad (A16)$$

$$K_\nu^{(m)}=\tfrac{1}{8}(p+p')_\nu\mu(p,p'), \quad (A17)$$

$$K_{\nu\sigma}^{(m)}=\frac{1}{8}\left[\frac{p_{3\nu}p_{3\sigma}}{p_3^2}\left[1-\frac{(p+p')^2}{4}\mu(p,p')\right]+\frac{(p+p')_\nu(p+p')_\sigma}{4}\mu(p,p')+\tfrac{1}{2}g_{\nu\sigma}\left[\ln\frac{m^2}{\Lambda^2}-\tfrac{3}{2}+\tfrac{1}{2}(p+p')^2\mu(p,p')\right]\right], \quad (A18)$$

$$p_3^2G_0^{(\mu)}=\frac{\kappa'}{4a'}\left[\ln2\kappa'\ln\frac{a'-\kappa'}{a'+\kappa'}+\Phi(a'-\kappa')-\Phi(-(a'+\kappa'))-\pi^2\right], \quad (A19)$$

$$p_3^2G_0^{(m)}=\frac{\kappa}{4a}\left[\ln2\kappa\ln\frac{a-\kappa}{a+\kappa}+\Phi(a-\kappa)-\Phi(-a-\kappa)-\pi^2\right], \quad (A20)$$

$$p_3^2G_\nu^{(\mu)}=\frac{\kappa'}{\kappa'+2}[q_\nu'(p_3^2G_0^{(\mu)}+\tfrac{1}{2}\ln2\kappa')+p_{3\nu}(2\mu^2G_0^{(\mu)}-\tfrac{1}{4}\ln2\kappa')], \quad (A21)$$

$$p_3^2G_\nu^{(m)}=\frac{\kappa}{\kappa+2}[p_\nu(p_3^2G_0^{(m)}+\tfrac{1}{2}\ln2\kappa)+p_{3\nu}(2m^2G_0^{(m)}-\tfrac{1}{4}\ln2\kappa)], \quad (A22)$$

where Φ is the Spence function defined in Eq. (A49).

C. Four-Denominator Integrals

The only four-denominator integrals appearing in our calculation are

$$I_{(0;\nu;\nu\mu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu;k_\nu k_\mu)d^4k}{(k^2-2pk)(k^2-2q'\cdot k)k^2(k-p_3)^2},$$

$$\hat{I}_{(0;\nu;\nu\mu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu;k_\nu k_\mu)d^4k}{(k^2-2pk)(k^2+2qk)k^2(k-p_3)^2}.$$

We shall leave the evaluation of the scalar functions, I_0 and $\hat{I}_0$, for later and consider first the tensor functions. We shall show that it is possible, by means of an algebraic technique given in Ref. 17, to reduce them to a combination of integrals of a lower tensor order. Let

$$p^{(1)}=p,\quad p^{(2)}=q',\quad p^{(3)}=p_3.$$

Then we write

$$I_\nu=\sum_{i=1}^{3}\alpha_ip^{(i)}, \quad (A23)$$

where the α_i are scalar functions of the $p^{(i)}$ and are to be determined. That this expansion is correct follows from the following argument: The $p^{(i)}$ span is a three-dimensional subspace. Let P be a vector in the direction perpendicular to this subspace. Then to evaluate the P component of I_ν we must restrict the region of integration to k along P. The integrand then becomes an odd function of k and must vanish upon being integrated.

Define

$$F_{(0;\nu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2pk)(k^2-2q'\cdot k)(k-p_3)^2}, \quad (A24)$$

$$H_{(0;\nu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2pk)(k^2-2q'\cdot k)k^2}. \quad (A25)$$

Then it is evident that

$$f_1=p^\nu\cdot I_\nu=\tfrac{1}{2}(F_0-G_0^{(\mu)})\,,$$
$$f_2=q'^\nu\cdot I_\nu=\tfrac{1}{2}(F_0-G_0^{(m)})\,,$$
$$f_3=p_3{}^\nu I_\nu=\tfrac{1}{2}(F_0-H_0+p_3{}^2I_0)\,,$$

where $G_0^{(\mu)}$ and $G_0^{(m)}$ are three-denominator integrals, already defined. It then follows from Eq. (A23) that

$$\alpha_i=\sum_{j=1}^{3}\Delta_{ij}{}^{-1}f_j\,, \tag{A26}$$

where

$$\Delta_{ij}=(p^{(i)}p^{(j)})=\begin{bmatrix}-m^2 & pq' & \tfrac{1}{2}p_3{}^2\\ pq' & -\mu^2 & \tfrac{1}{2}p_3{}^2\\ \tfrac{1}{2}p_3{}^2 & \tfrac{1}{2}p_3{}^2 & p_3{}^2\end{bmatrix}$$

and Δ^{-1} is defined by the condition that $\Delta^{-1}\Delta=1$. It can be shown that

$$\Delta^{-1}=\frac{-1}{D}\begin{bmatrix}(2\mu^2+\tfrac{1}{2}p_3{}^2) & (2(pq')-\tfrac{1}{2}p_3{}^2) & -(\mu^2+pq')\\ (2(pq')-\tfrac{1}{2}p_3{}^2) & (2m^2+\tfrac{1}{2}p_3{}^2) & -(m^2+pq')\\ -(\mu^2+pq') & -(m^2+pq') & (2/p_3{}^2)[(pq')^2-m^2\mu^2]\end{bmatrix},$$

where

$$D=2[m^2\mu^2-(pq')^2]+\tfrac{1}{2}p_3{}^2(m^2+\mu^2+2pq')\,. \tag{A27}$$

Then Eq. (A26) for $i=1$, 2 gives

$$\alpha_1=(1/2D)[(\mu^2+pq')(p_3{}^2I_0-2H_0)+(2\mu^2+\tfrac{1}{2}p_3{}^2)G_0^{(\mu)}+(2pq'-\tfrac{1}{2}p_3{}^2)G_0^{(m)}]\,, \tag{A28}$$

$$\alpha_2=(1/2D)[(m^2+pq')(p_3{}^2I_0-2H_0)+(2m^2+\tfrac{1}{2}p_3{}^2)G_0^{(m)}+(2pq'-\tfrac{1}{2}p_3{}^2)G_0^{(\mu)}]\,. \tag{A29}$$

We have not listed α_3 because it isn't needed in the calculation. We have made use of the result that

$$F_0=H_0\,. \tag{A30}$$

To see this, first let $k\to k+p_3$ in F_0 [Eq. (A24)]; this shows that F_0 is the same scalar function of p' and q as H_0 is of p and q'. However, p^2, pq', and q'^2 are the only scalars we can form from p and q' and these are invarient to the substitution $p\to p'$, $q'\to q$. Q. E. D.

Let us now apply the same procedure to $I_{\nu\mu}$. We may write

$$I_{\nu\mu}=\alpha_{ij}p_\nu{}^{(i)}p_\mu{}^{(j)}+\epsilon g_{\mu\nu}\,, \tag{A31}$$

where α_{ij} and ϵ are scalar functions of the $p^{(i)}$. Let

$$f_\nu{}^{(1)}=p^\sigma I_{\sigma\nu}=\tfrac{1}{2}(F_\nu-G_\nu{}^{(\mu)})\,,$$
$$f_\nu{}^{(2)}=q'^\sigma I_{\sigma\nu}=\tfrac{1}{2}(F_\nu-G_\nu{}^{(m)})\,,$$
$$f_\nu{}^{(3)}=p_3{}^\sigma I_{\sigma\nu}=\tfrac{1}{2}(F_\nu-H_\nu+p_3{}^2I_\nu)\,.$$

We now further expand

$$f_\nu{}^{(i)}=\beta_{ij}p_\nu{}^{(j)}\,, \tag{A32}$$

where we suppose the β_{ij} to be given and we wish to solve for α_{ij} and ϵ. Multiplying Eq. (A31) by $p_\nu{}^{(k)}p_\mu{}^{(m)}$ and contracting on ν and μ we get

$$\alpha_{ij}\Delta_{ki}\Delta_{mj}+\epsilon\Delta_{km}=\beta_{ml}\Delta_{kl}\,,$$

which gives

$$\alpha_{ij}=\sum_{k=1}^{3}\Delta_{jk}{}^{-1}[\beta_{ki}-\epsilon\delta_{ki}]\,. \tag{A33}$$

ϵ is to be obtained from the condition $I_\nu{}^\nu=F$, which gives

$$\epsilon=F-\sum_k\beta_{kk}\,. \tag{A34}$$

The integral I_0 contains an infrared divergence. I_0, however, appears in our calculation only in the combination $p_3{}^2I_0-2H_0$ which is divergentless. By Eq. (A30) we write

$$p_3{}^2I_0-2H_0=\frac{i}{(2\pi)^2}\int\frac{d^4k(2p_3k-2k^2)}{(k^2-2pk)(k^2-2q'k)k^2(k-p_3)^2}\,.$$

It is this integral we now propose to evaluate. Using Eqs. (A1)–(A3), (A5) we get

$$p_3{}^2I_0-2H_0=-\frac{1}{2}\int_0^1\int_0^1 xdxdy\times\int_0^1\frac{z(zp_x{}^2-2\Delta_x+p_3p_x)dz}{(zp_x{}^2-\Delta_x)^2}\,,$$

where $p_x=xp_y+(1-x)p_3$, $\Delta_x=p_3{}^2(1-x)$, and

$$p_y=yp+(1-y)q'\,. \tag{A35}$$

We do the z integration as follows:

$$\int_0^1\frac{z(zp_x{}^2-2\Delta_x+p_3p_x)dz}{(zp_x{}^2-\Delta_x)^2}$$
$$=\int_0^1\frac{zdz}{zp_x{}^2-\Delta_x}+(p_3p_x-\Delta_x)\int_0^1\frac{zdz}{(zp_x{}^2-\Delta_x)^2},$$
$$=\frac{1}{p_x{}^2}\left(\frac{p_x{}^2-p_3p_x}{p_x{}^2-\Delta_x}\right)+\frac{p_3p_x}{(p_x{}^2)^2}\ln\left(\frac{\Delta_x-p_x{}^2}{\Delta_x}\right),$$
$$=\frac{1}{2p_y{}^2x}\left(\frac{2p_y{}^2x-p_3{}^2}{p_y{}^2x^2+p_3{}^2(1-x)}\right)+\frac{p_3{}^2(2-x)\ln[-p_y{}^2x^2/p_3{}^2(1-x)]}{2[p_y{}^2x^2+p_3{}^2(1-x)]^2}\,.$$

In the last step we have used the relations

$$p_x^2=p_y^2x^2+p_3^2(1-x)\,,$$
$$p_3\cdot p_x=\tfrac{1}{2}p_3^2(2-x)\,,$$
$$p_3\cdot p_y=\tfrac{1}{2}p_3^2\,.$$

We can therefore write

$$p_3^2I_0-2H_0=-\frac{1}{4}\int_0^1 dy(\mathrm{I}+\mathrm{II})\,,$$

where

$$\mathrm{I}=\frac{1}{p_y^2}\int_0^1\frac{(2p_y^2x-p_3^2)dx}{p_y^2x^2+p_3^2(1-x)}=\frac{1}{p_y^2}\ln\left|\frac{p_y^2}{p_3^2}\right|,$$

$$\mathrm{II}=\int_0^1\frac{p_3^2x(2-x)\ln[-p_y^2x^2/p_3^2(1-x)]}{[p_y^2x^2+p_3^2(1-x)]^2}dx$$
$$=-\frac{1}{p_y^2}\int_0^\infty\frac{dt\ln t}{(1-t)^2}=0\,.$$

In II we have made the change of variable to $t=-p_y^2x^2/p_3^2(1-x)$. The last step is gotten by letting $t\rightarrow 1/t$, then $\mathrm{II}\rightarrow-\mathrm{II}$ and must therefore be zero. In both I and II we have ignored the imaginary contributions from the pole at x_0, where $p_y^2x_0^2+p_3^2(1-x_0)=0$. This, however, introduces no error in I+II, since the residue of the sum of the integrands of I+II, which is given by

$$\lim_{x\to x_0}\left\{2p_y^2x-p_3^2+\frac{p_y^2x(2-x)}{1-x}\left[\lim_{t\to 1}\left(\frac{\ln t}{1-t}\right)\right]\right\},$$

is equal to

$$\lim_{x\to x_0}\left[\frac{-p_y^2x^2+p_3^2(1-x)}{1-x}\right]=0\,.$$

Therefore,

$$p_3^2I_0-2H_0=-\frac{1}{4}\int_0^1\frac{dy}{p_y^2}\ln\left|\frac{p_y^2}{p_3^2}\right|. \tag{A36}$$

From Eq. (A35) we get

$$p_y^2=\frac{d^2}{(p-q')^2}\left\{\left[\frac{(p-q')^2y+q'(p-q')}{d}\right]^2-1\right\}, \tag{A37}$$

where

$$d^2=(pq')^2-m^2\mu^2>0\,.$$

Since p_y is the sum of two timelike vectors, it too must be timelike, i.e., $p_y^2<0$. We then have the following two cases:

Case a. $(p-q')^2<0$. It then follows from Eq. (A37) that

$$\left|\frac{(p-q')^2y+q'(p-q')}{d}\right|>1\,.$$

So let

$$\coth\theta=[(p-q')^2y+q'(p-q')]/d\,.$$

Then

$$p_3^2I_0-2H_0=\frac{1}{4d}\int_a^b\ln\left[\frac{-d^2\,\mathrm{csch}^2\theta}{(p-q')^2p_3^2}\right]d\theta\,, \tag{A38}$$

where

$$a=\coth^{-1}\left[\frac{q'(p-q')}{d}\right]\simeq-\tfrac{1}{2}\ln\left[\frac{\mu^2+2pq'}{\mu^2}\right],$$

$$b=\coth^{-1}\left[\frac{p(p-q')}{d}\right]\simeq\tfrac{1}{2}\ln\left[\frac{4(pq')^2}{m^2(\mu^2+2pq')}\right].$$

We do the θ integration by letting $t=e^{-2\theta}$, i.e.,

$$\int_a^b\ln(\mathrm{csch}\theta)^2d\theta=-2\int_a^b\ln(\sinh\theta)d\theta$$
$$=\int_{e^{-2a}}^{e^{-2b}}\frac{dt}{t}[\ln(1-t)-\ln 2-\tfrac{1}{2}\ln t]\,,$$

and there results

$$p_3^2I_0-2H_0=\frac{1}{4d}\left[(b-a)\ln\left(\frac{-4d^2}{(p-q')^2p_3^2}\right)+a^2-b^2+\Phi(e^{-2a})-\Phi(e^{-2b})\right], \tag{A39}$$

where $\Phi(x)$ is the Spence function defined in Eq. (A49). If we use exact values for a and b then this expression is exact. The approximate expressions for a and b come from neglecting m/μ compared with unity and neglecting unity compared to $-p\cdot q'/m\mu$, i.e., $-p\cdot q'/m\mu\gg 1\gg m/\mu$. Applying this approximation to Eq. (A39) we get

$$p_3^2I_0-2H_0\simeq-\frac{1}{4pq'}\left[\ln\left(\frac{-2pq'}{m\mu}\right)\ln\left(\frac{-p\cdot q'}{m\mu\kappa}\right)+\Phi\left(1+\frac{2pq'}{\mu^2}\right)-\Phi\left(\frac{(\mu^2+2pq')m^2}{(2pq')^2}\right)\right]. \tag{A40}$$

Case b. $(p-q')^2>0$. By Eq. (A37) we can let

$$\tanh\theta=[(p-q')^2y+q'(p-q')]/d\,.$$

This leads to

$$p_3^2I_0-2H_0=\frac{1}{4d}\left[(b'-a')\ln\left(\frac{4d^2}{(p-q')^2p_3^2}\right)+a'^2-b'^2+\Phi(-e^{-2a'})-\Phi(-e^{-2b'})\right],$$

where

$$a'=\tanh^{-1}[q'(p-q')/d]\,,$$
$$b'=\tanh^{-1}[p(p-q')/d]\,.$$

Making the same approximations as in case a we obtain again Eq. (A38).

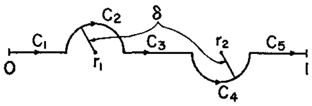

FIG. 6. Path of integration for the evaluation of $p_3^2\hat{I}_0-2\hat{H}_0$.

Instead of $\hat{I}_0$, the integral pertinent to our calculation is $p_3^2\hat{I}_0-2\hat{H}_0$, where

$$\hat{H}_0=\frac{i}{(2\pi)^4}\int\frac{d^4k}{(k^2-2pk)(k^2+2qk)k^2}. \qquad (A41)$$

It is clear that all the arguments used to derive Eq. (A36) apply equally well to $p_3^2\hat{I}_0-2\hat{H}_0$. Hence,

$$p_3^2\hat{I}_0-2\hat{H}_0=-\frac{1}{4}\int_0^1\frac{dy}{\hat{p}_y^2+i\delta}\ln\left|\frac{\hat{p}_y^2}{p_3^2}\right|, \qquad (A42)$$

where now

$$\hat{p}_y=py-q(1-y)$$

and we have carried along the small negative imaginary additions to m and μ, producing the term $+i\delta$ in the denominator. We write

$$\hat{p}_y^2=[\hat{d}^2/(p+q)^2][(h(y))^2-1],$$

where $\hat{d}^2=(pq)^2-m^2\mu^2$ and

$$h(y)=[(p+q)^2y-q(p+q)]/\hat{d}.$$

It is no longer true that $\hat{p}_y^2<0$, in fact it is quite clear that $\hat{p}_y^2$ has two roots in the interval (0,1). Let r_1 and r_2 be these roots and let $r_1<r_2$. Then

$$\hat{p}_y^2=(p+q)^2(y-r_1)(y-r_2)$$

and

$$r_2-r_1=-2\hat{d}/(p+q)^2.$$

We can break up the integration region into five parts as shown in Fig. 6. The contours c_2 and c_4 are so chosen as to give $\hat{p}_y^2$ an imaginary part greater than zero. We first do the integration for regions c_1, c_3, c_5. In c_1 and c_5, $|h(y)|>1$ so let $\coth\theta=h(y)$, whereas in c_3, $|h(y)|<1$ so let

$$\tanh\theta=h(y).$$

Then

$$-\frac{1}{4}\left[\int_0^{r_1-\delta}+\int_{r_1+\delta}^{r_2-\delta}+\int_{r_2+\delta}^{1}\right]\frac{dy}{\hat{p}_y^2}\ln\left|\frac{\hat{p}_y^2}{p_3^2}\right|$$

$$=\frac{1}{4\hat{d}}\left[\int_{\hat{a}}^{\hat{b}}\ln\left[\frac{-\hat{d}^2\operatorname{csch}^2\theta}{(p+q)^2p_3^2}\right]d\theta+\int_{-\infty}^{\infty}\ln(\coth\theta)^2d\theta\right],$$

where

$$\hat{a}=\coth^{-1}[-q(p+q)/\hat{d}]\simeq\tfrac{1}{2}\ln[1-(2pq/\mu^2)],$$
$$\hat{b}=\coth^{-1}[p(p+q)/\hat{d}]\simeq\tfrac{1}{2}\ln[m^2(\mu^2-2pq)/(2pq)^2].$$

The first integral on the right is essentially the same as Eq. (A38). It is therefore equal to Eq. (A39) with $q'\to -q$. The second integral

$$\int_{-\infty}^{\infty}\ln(\coth\theta)^2d\theta=-2\int_{-\infty}^{\infty}\left[\ln(\tanh\theta)d\theta\right]d\theta,$$

$$=-2\int_{-1}^{1}\frac{\ln x}{1-x^2}dx=\tfrac{1}{2}\pi^2.$$

The integration for c_2 is done by letting $y=r_1-\delta e^{-i\theta}$ then

$$-\frac{1}{4}\int\frac{dy}{p_y^2}\ln\frac{p_y^2}{p_3^2}$$

$$=\frac{i}{4(q+p)^2(r_1-r_2)}\int_0^{\pi}\ln\left[\frac{(p+q)^2(r_1-r_2)\delta e^{i\theta}}{p_3^2}\right]d\theta,$$

$$=\frac{\pi^2}{8(p+q)^2(r_1-r_2)}+\text{(imaginary part)}.$$

For c_4 let $y=r_2+\delta e^{i(\theta-\pi)}$. Then the contribution from c_2 and c_4 together is $(1/4\hat{d})(\pi^2/2)$. Upon combining all the contributions we arrive at

$$p_3^2\hat{I}_0-2\hat{H}_0=\frac{1}{4\hat{d}}\left[(\hat{b}-\hat{a})\ln\left(\frac{-4\hat{d}}{(p+q)^2p_3^2}\right)\right.$$
$$\left.+\Phi(e^{-2\hat{a}})-\Phi(e^{-2\hat{b}})+\hat{a}^2-\hat{b}^2+\pi^2\right]. \qquad (A43)$$

If we now make the high-energy approximation, i.e.,

$$-pq/m\mu\gg 1\gg m/\mu,$$

we get

$$p_3^2\hat{I}_0-2\hat{H}_0=\frac{1}{4pq}\left[\ln\frac{-2pq}{m\mu}\ln\frac{-pq}{m\mu\kappa}+\Phi\left(\frac{\mu^2-2pq}{\mu^2}\right)\right.$$
$$\left.-\Phi\left(\frac{m^2(\mu^2-2pq)}{(2pq)^2}\right)-\pi^2\right]. \qquad (A44)$$

Comparing Eq. (A44) with Eq. (A40) we see that the interchange $q\leftrightarrow -q'$ in Eq. (A40) produces Eq. (A44). This interchange leads to a term $(\ln-1)^2$ which corresponds exactly to the $-\pi^2$ term in Eq. (A44) if we keep only the real part.

D. Miscellaneous Integrals

We have given the results of some of the previous integrations in terms of the functions $K(p_a,p_b)$ and $\mu(p_a,p_b)$ defined in Eqs. (II.10), (II.11), respectively. $K(p_a,p_b)$ contains the infrared divergence and completely cancels out in our calculation. We will, however, be interested in $K(p,-q)$, $K(p',-q')$, in terms of $K(p,q)$, $K(p',q')$ [cf., Eq. (II.24)]. Comparing Eq. (A36) with Eq. (II.10) we readily see that $K(p,-q)/4(pq)$ can be obtained from Eq. (A44) by the substitution $\kappa\to\lambda^2/2m^2$. Furthermore, the substitution $\kappa\to\lambda^2/2m^2$ and $q'\to q$ carries Eq. (A39) into $-K(p,q)/4(pq)$. We can therefore write

$$\mathrm{Re}K(p,-q)=K(p,q)+\tfrac{1}{2}\delta_\kappa,$$

where

$$\delta_\kappa=2\left[\Phi\left(\frac{(\mu^2-2pq)}{\mu^2}\right)-\Phi\left(\frac{\mu^2+2pq}{\mu^2}\right)-\Phi\left(\frac{m^2(\mu^2-2pq)}{(2pq)^2}\right)+\Phi\left(\frac{m^2(\mu^2+2pq)}{(2pq)^2}\right)-\pi^2\right]. \quad \text{(A45)}$$

It is easy to see that $K(p',q')=K(p,q)$ and $K(p',-q')=K(p,-q)$. This follows directly from Eq. (A30) and the analogous relation $\hat{F}_0=\hat{H}_0$. Alternatively, it can be seen by examining the $p_y{}^2$ terms entering each side of the equality: They are equal. Hence,

$$\mathrm{Re}[K(p,-q)+K(p',-q')]=K(p,q)+K(p',q')+\delta_\kappa. \quad \text{(A46)}$$

We next evaluate $\mu(p_a,p_b)$.

$$\mu(p_a,p_b)=\int_0^1\frac{dy}{p_y{}^2}$$

and $p_y=p_ay+p_b(1-y)$. We may write $p_y{}^2=(p_a-p_b)^2\times(y+r_1)(y+r_2)$, where

$$r_{1,2}=[p_b(p_a-p_b)\pm d]/(p_a-p_b)^2$$

and

$$d^2=(p_ap_b)^2-m_a{}^2m_b{}^2. \quad \text{(A47)}$$

It can then be show that

$$\mu(p_a,p_b)=-\frac{1}{2d}\ln\frac{(1+r_1)r_2}{(1+r_2)r_1}=-\frac{1}{2d}\ln\left[\frac{(p_ap_b)-d}{(p_ap_b)+d}\right]. \quad \text{(A48)}$$

E. Spence Function

The Spence function[19] $\Phi(x)$ is defined by

$$\Phi(x)=-\int_0^x\frac{\ln|1-u|}{u}du. \quad \text{(A49)}$$

The following is a list of relations for the Spence function which enter this problem. The derivation of those relations together with a numerical tabulation of $\Phi(x)$ are given in Ref. 19.

$$\Phi(1)=\tfrac{1}{6}\pi^2, \quad \text{(A50)}$$

$$\Phi(-1)=-\tfrac{1}{12}\pi^2, \quad \text{(A51)}$$

[19] K. Mitchell, Phil. Mag. 40, 351 (1949).

$$\Phi(x)+\Phi(1/x)=2\Phi(1)-\tfrac{1}{2}\ln^2x;\quad (x\geq 0) \quad \text{(A52)}$$

$$\Phi(x)+\Phi(1/x)=2\Phi(-1)-\tfrac{1}{2}\ln^2(-x);\quad (x\leq 0) \quad \text{(A53)}$$

$$\Phi(x)+\Phi(1-x)=\Phi(1)-\ln x\ln(1-x);\quad (0\leq x\leq 1) \quad \text{(A54)}$$

$$\Phi(1-1/x)+\Phi(1-x)=-\tfrac{1}{2}\ln^2x. \quad \text{(A55)}$$

APPENDIX B: ULTRAVIOLET INTEGRALS

In this section we give the details for the evaluation of those matrix elements which suffer an ultraviolet divergence. The usual renormalization techniques are used. We will illustrate these techniques for "scalar" electrodynamics and more or less quote the results for "spinor" electrodynamics, since the latter case is well known and can be readily found in the literature.[8]

A. Meson Vertex Function

e^2 corrections to the single-meson corner are shown in Fig. 7. That is, for any diagram containing a single-meson corner (say M_1) some of its corrections (M_{8-10})

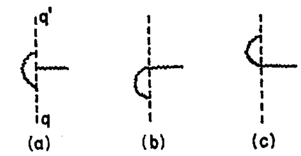

FIG. 7. Scalar vertex diagrams.

can be gotten by simply replacing $(q+q')_\nu$ of that corner by $(\alpha/2\pi)(\Lambda_\nu{}^{(a)}+\Lambda_\nu{}^{(b)}+\Lambda_\nu{}^{(c)})$, where from the rules of correspondence we get

$$\Lambda_\nu{}^{(a)}=\frac{-2i}{(2\pi)^2}\int\frac{(2q-k)\cdot(2q'-k)(q_\nu+q_\nu'-2k_\nu)d^4k}{(k^2-2qk)(k^2-2q'\cdot k)k^2},$$

$$\Lambda_\nu{}^{(b)}=\frac{4i}{(2\pi)^2}\int\frac{(2q_\nu'-k_\nu)d^4k}{(k^2-2q'k)k^2}, \quad \text{(B1)}$$

$$\Lambda_\nu{}^{(c)}=\frac{4i}{(2\pi)^2}\int\frac{(2q_\nu-k_\nu)d^4k}{(k^2-2q\cdot k)k^2}.$$

If we let

$$\Lambda_\nu{}^{(\mu)}\equiv\Lambda_\nu{}^{(a)}+\Lambda_\nu{}^{(b)}+\Lambda_\nu{}^{(c)}=\frac{-2i}{(2\pi)^2}\int\frac{N_\nu{}^{(\mu)}d^4k}{(k^2-2q\cdot k)(k^2-2q'k)k^2}, \quad \text{(B2)}$$

then

$$N_\nu{}^{(\mu)}=4(qq')Q_\nu-8(qq')k_\nu-2(Q\cdot k)Q_\nu+8(q\cdot k)q_\nu'+8(q'\cdot k)q_\nu+2k^2k_\nu-3k^2Q_\nu.$$

We can therefore write

$$\Lambda_\nu{}^{(\mu)}=-2\{4(qq')K_0{}^{(\mu)}Q_\nu-8(qq')K_\nu{}^{(\mu)}-2(Q^\sigma K_\sigma{}^{(\mu)})Q_\nu+8(q^\sigma K_\sigma{}^{(\mu)})q_\nu'-8(q'^\sigma K_\sigma{}^{(\mu)})q_\nu+2L_\nu{}^{(\mu)}-3L_0{}^{(\mu)}Q_\nu\}, \quad \text{(B3)}$$

here

$$K_{(0;\nu)}{}^{(\mu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2qk)(k^2-2q'k)k^2},\tag{B4}$$

$$L_{(0;\nu)}{}^{(\mu)}=\frac{i\Lambda^2}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2qk)(k^2-2q'k)(k^2+\Lambda^2)}.\tag{B5}$$

We have included the regulator $\Lambda^2/(k^2+\Lambda^2)$ in Eq. (B5) which is to be considered in the limit of $\Lambda^2\to\infty$. These integrals are evaluated by means of the usual Feynman[18] technique of combining denominators, and the results are given in Eqs. (A7), (A9), (A14), and (A16).

Upon substituting these results into Eq. (B3) we get

$$\Lambda_\nu{}^{(\mu)}(q,q')=-Q_\nu\left[K(q,q')-2(qq')\mu(q,q')+\ln\frac{\Lambda^2}{\mu^2}+\frac{5}{4}\right].\tag{B6}$$

In addition to the infrared divergence contained in $K(q,q')$ we have as expected the logarithmically divergent term $\ln(\Lambda^2/\mu^2)$; where $\Lambda^2\to\infty$. This is the ultraviolet divergence. The prescription for its removal is to subtract

$$\tfrac{1}{2}[\Lambda_\nu{}^{(\mu)}(q,q)+\Lambda_\nu{}^{(\mu)}(q',q')]=-Q_\nu\left[K(q,q)-2+\ln\frac{\Lambda^2}{\mu^2}-\frac{5}{4}\right].\tag{B7}$$

Since $K(q,q)$ is a constant, this is tantamount to a redefinition of the meson charge e; hence a charge renormalization. We obtain for the finite vertex function

$$\Lambda_{F\nu}{}^{(\mu)}=-Q_\nu[K(q,q')-K(q,q)-2(qq')\mu(q,q')+2].\tag{B8}$$

We will now show that charge renormalization is in fact a "spurious" charge renormalization; i.e., the infinite term, Eq. (B7), is completely cancelled by a corresponding term from the self-energy diagrams. So consider the meson self-energy diagrams M_{11-14}. It is clear that the infinities arising from the balloon diagrams, M_{13} and M_{14}, are completely removed by mass renormalization, since these terms are proportional to $\int d^4k/k^2$ and don't contain external momenta as variables. Hence all we need consider is M_{11} and M_{12}. From the rules of correspondence we have

$$M_{11}=\frac{\alpha}{2\pi}\frac{\Sigma(q')}{q'^2+\mu^2}M_1,\qquad M_{12}=\frac{\alpha}{2\pi}\frac{\Sigma(q)}{q^2+\mu^2}M_1,\tag{B9}$$

where

$$\Sigma(q)=\frac{-2i}{(2\pi)^2}\int\frac{d^4k(2q-k)^2}{k^2[(q-k)^2+\mu^2]}.\tag{B10}$$

We can expand (the regulated) $\Sigma(q)$ as follows:

$$\Sigma(q)=A+B(q^2+\mu^2)+(q^2+\mu^2)^2\Sigma_F,\tag{B11}$$

where A and B are "infinite" constants and Σ_F is finite. A can be removed by a redefinition of mass and B by a redefinition of charge. Using the same arguments as are used in the electron self-energy case, it can be shown that

$$\left.\frac{\Sigma(q)-A}{q^2+\mu^2}\right|_{q^2=-\mu^2}=\tfrac{1}{2}B,$$

and therefore after mass renormalization we get

$$M_{11}+M_{12}=(\alpha/2\pi)BM_1.$$

To prove that we have a "spurious" charge renormalization we need show that

$$Q_\nu B=-\tfrac{1}{2}[\Lambda_\nu{}^{(\mu)}(q,q)+\Lambda_\nu{}^{(\mu)}(q',q')].\tag{B12}$$

Differentiating Eq. (B10) with respect to q^ν we get

$$\frac{\partial\Sigma(q)}{\partial q^\nu}=\frac{-2i}{(2\pi)^2}\left[\int\frac{4(2q_\nu-k_\nu)d^4k}{k^2(k^2-2qk)}-2\int\frac{(2q-k)^2(q_\nu-k_\nu)d^4k}{k^2(k^2-2qk)^2}\right]=-\Lambda_\nu{}^{(\mu)}(q,q).$$

But from Eq. (B11) we have

$$\partial\Sigma(q)/\partial q^\nu=2q_\nu B.$$

Hence Eq. (B12) follows.

B. Electron Vertex Function

If in M_1 we replace γ_ν by $(\alpha/2\pi)\Lambda_\nu{}^{(m)}$, where

$$\Lambda_\nu{}^{(m)}=\frac{2i}{(2\pi)^2}\int\frac{N_\nu{}^{(m)}d^4k}{(k^2-2pk)(k^2-2p'k)k^2}\tag{B13}$$

and

$$N_\nu{}^{(m)}=\gamma^\rho[i(p'-k)-m]\gamma_\nu[i(p-k)-m]\gamma_\rho,\tag{B14}$$

we get M_5. It can be show that

$$\Lambda_\nu{}^{(m)}(p,p')\doteq-\gamma_\nu\left[K(p,p')-(\tfrac{1}{4}p_3{}^2+2pp')\mu(p,p')+\tfrac{1}{2}\ln\frac{m^2}{\Lambda^2}-\frac{1}{4}\right]+\frac{m}{4i}\mu(p,p')[p_3,\gamma_\nu].\tag{B15}$$

To remove the ultraviolet divergence we subtract $\Lambda_\nu{}^{(m)}(p,p)$ and result with the finite expression

$$\Lambda_{F\nu}{}^{(m)}\doteq-\gamma_\nu[K(p,p')-K(p,p)-(\tfrac{1}{4}p_3{}^2+2pp')\mu(p,p')+2]+\Lambda_\nu{}^{(A)},\tag{B16}$$

where

$$\Lambda_\nu{}^{(A)}=(m/4i)\mu(p,p')(p_3\gamma_\nu-\gamma_\nu p_3).\tag{B17}$$

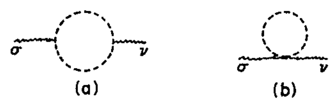

FIG. 8. Scalar vacuum polarization diagrams.

$\Lambda_\nu^{(A)}$ gives rise to the anomalous magnetic moment and is negligible in our calculation.[20]

C. Meson Vacuum Polarization

The e^2 corrections to the photon propagator arising from "scalar" electrodynamics are depicted in Fig. 8. It will become apparent that since diagram (b) is independent of k it contributes nothing and can therefore be ignored. So if

$$\Pi_{\sigma\nu}^{(\mu)}=\frac{-2i}{(2\pi)^2}\int\frac{(2\hat{q}-k)_\sigma(2\hat{q}-k)_\nu d^4\hat{q}}{(\hat{q}^2+\mu^2)[(\hat{q}-k)^2+\mu^2]}, \quad \text{(B18)}$$

then the corrections represented by diagram (a) can be obtained by simply substituting in the photon propagator

$$g_{\sigma\nu}\longrightarrow(\alpha/2\pi)[\Pi_{\sigma\nu}^{(\mu)}(k)/k^2]. \quad \text{(B19)}$$

By employing the conditions of relativistic and gauge invariance it can be formally shown that[21]

$$\Pi_{\sigma\nu}^{(\mu)}(k)=-\frac{(k_\sigma k_\nu-g_{\sigma\nu}k^2)\Pi_\rho^{(\mu)\rho}(k)}{3k^2} \quad \text{(B20)}$$

and

$$\Pi_\rho^{(\mu)\rho}(0)=0. \quad \text{(B21)}$$

This last condition is clearly not satisfied, since Eq. (B18) gives

$$\Pi_\rho^{(\mu)\rho}(0)=\frac{-2i}{(2\pi)^2}\int\frac{(2\hat{q})^2d^4\hat{q}}{(\hat{q}^2+\mu^2)^2},$$

which is quadratically divergent. We therefore redefine

$$\Pi_\rho^{(\mu)\rho}(k)=\frac{-2i}{(2\pi)^2}\left[\int\frac{(2\hat{q}-k)^2d^4\hat{q}}{(\hat{q}^2+\mu^2)[(\hat{q}-k)^2+\mu^2]}-\int\frac{(2\hat{q})^2d^4\hat{q}}{(\hat{q}^2+\mu^2)^2}\right], \quad \text{(B22)}$$

which satisfies Eq. (B21) and is logarithmically divergent. It is because of this subtraction that diagram (b) of Fig. 8 contributes nothing. We can now write

$$(3k^2)^{-1}\Pi_\rho^{(\mu)\rho}(k)=A+\Pi_F^{(\mu)}(k^2),$$

where A is an infinite constant independent of k and $\Pi_F^{(\mu)}(k^2)$ is finite and goes to zero with k. We then subtract A and we have from Eq. (B20)

$$\Pi_{\sigma\nu}^{(\mu)}(k)=-(k_\sigma k_\nu-g_{\sigma\nu}k^2)\Pi_F^{(\mu)}(k^2). \quad \text{(B23)}$$

The term proportional to $k_\sigma k_\nu$ contributes nothing, since it is either contracted with γ^ν and employing the δ function it gives rise to $p_3\doteq 0$, or it is contracted with Q^σ and employing the δ function it leads to $(q+q')(q-q')=0$. Hence we may write

$$\Pi_{\sigma\nu}^{(\mu)}(k)=g_{\sigma\nu}k^2\Pi_F^{(\mu)}(k^2) \quad \text{(B24)}$$

and the substitution of Eq. (B19) becomes

$$g_{\sigma\nu}\longrightarrow(\alpha/2\pi)g_{\sigma\nu}\Pi_F^{(\mu)}(k^2). \quad \text{(B25)}$$

We now return to the evaluation $\Pi_F^{(\mu)}(p_3^2)$. If in Eq. (B22) we let $k=p_3$ and $\hat{q}=l-q$, we get

$$\Pi_\rho^{(\mu)\rho}(p_3)=2[p_3^2L_0^{(\mu)}-4p_3^\sigma L_\sigma^{(\mu)}+8\mu^2p_3^\sigma S_\sigma^{(\mu)}],$$

where $L_{(0;\nu)}^{(\mu)}$ are defined in Eq. (B5) and

$$S_\sigma^{(\mu)}=\frac{i}{(2\pi)^2}\int\frac{l_\sigma d^4l}{(l^2-2lq)^2(l^2-2lq')}. \quad \text{(B26)}$$

From Eqs. (A7) ,(A9), and (A12)

$$(3p_3^2)^{-1}\Pi_\rho^{(\mu)\rho}(p_3)=\frac{1}{3}\left[\ln\frac{\Lambda}{\mu}+\tfrac{1}{2}-\frac{Q^2}{4}\mu(q,q')+\frac{4\mu^2}{p_3^2}\left[1-\frac{Q^2}{4}\mu(q,q')\right]\right]$$

and hence

$$\Pi_F^{(\mu)}(p_3^2)=\frac{Q^2}{12}\mu(q,q')-\frac{4\mu^2}{3p_3^2}\left[1-\frac{Q^2}{4}\mu(q,q')\right]-\frac{4}{9}. \quad \text{(B27)}$$

This can be rewritten in a more calculable form by using Eq. (A48);

$$\Pi_F^{(\mu)}(p_3^2)=\frac{(\kappa'+2)^2}{6a'\kappa'}\ln\frac{a'+\kappa'}{a'-\kappa'}-\frac{4}{9}-\frac{2}{3\kappa'}. \quad \text{(B28)}$$

D. Electron Vacuum Polarization

For this case

$$\Pi_{\sigma\nu}=\frac{-2i}{(2\pi)^2}\int\frac{d^4\hat{p}\,\mathrm{Tr}\{\gamma_\sigma(i\hat{p}-m)\gamma_\nu(i(\hat{p}-k)-m)\}}{(\hat{p}^2+m^2)[(\hat{p}-k)^2+m^2]}. \quad \text{(B29)}$$

After imposing the requirement of Eq. (B21) we have

$$\Pi_\nu^\nu(k)=\frac{-2i}{(2\pi)^2}\int\left[\frac{(\hat{p}^2+2m^2-\hat{p}\cdot k)}{(\hat{p}^2+m^2)[(\hat{p}-k)^2+m^2]}-\frac{\hat{p}^2+2m^2}{(\hat{p}^2+m^2)^2}\right]d^4\hat{p}. \quad \text{(B30)}$$

It can be shown that

$$\frac{\Pi_\nu^\nu(-p_3)}{3p_3^2}=\frac{4}{3}\left[\ln\frac{m}{\Lambda}-\frac{1}{2}+\frac{(p+p')^2}{4}\mu(p,p')+(2m^2/p_3^2)(1-\tfrac{1}{4}(p+p')^2\mu(p,p'))\right]$$

[20] Cf., Eq. (6) of Ref. 6, and subsequent discussion.
[21] Cf., Sec. 9-5 of Ref. 8, for example.

and finally

$$\Pi_F(p_3^2)=\tfrac{1}{3}(p+p')^2\mu(p,p')+(2m^2/p_3^2)[1-\tfrac{1}{4}(p+p')^2\mu(p,p')]-(10/9).$$

APPENDIX C: TWO-PHOTON EXCHANGE CALCULATION

The contributions to the cross section arising from the two-photon exchange diagrams, M_2, M_3, and M_4, are derived in this appendix. In Sec. II it is shown how we can extract from J_2 and J_3 [Eqs. (II.16), (II.17)] their infrared parts. What remains is divergentless and is called J_2^0 and J_3^0, respectively [cf., Eqs. (II.20), (II.22)]. It is the cross section $d\sigma_J$ arising from $J^0=J_2^0+J_3^0+J_4$ that we wish to calculate here. By taking account of the appearance of J^0 between the spinors $\bar{u}(\mathbf{p}')$ and $u(\mathbf{p})$, it is possible to express all slashed quantities in J^0 in terms of Q(say). We can therefore write

$$J^0\doteq(\alpha/\pi)(A+QB), \tag{C1}$$

where A and B contain no γ matrices, from which we get

$$\sum_{\text{spins}} M_1^\dagger M_{2-4}=\frac{\alpha}{\pi}p_3^2\left[B+A\frac{\mathrm{Tr}[(ip-m)Q(ip'-m)]}{\mathrm{Tr}[(ip-m)Q(ip'-m)Q]}\right]\sum_{\text{spins}} M_1^\dagger M_1.$$

It then follows that

$$d\sigma_J=(\alpha/\pi)d\sigma_0\delta_J, \tag{C2}$$

where

$$\delta_J=2p_3^2\,\mathrm{Re}\{B+[2im(p\cdot Q)A/T_0]\}. \tag{C3}$$

We now proceed to obtain expressions for A and B. By expanding the numerator of the integrand of J_3, Eq. (II.17), we can show that

$$J_3^0\doteq(-\alpha/\pi)\{4Q(pq')I_0-2[2(pq')\gamma^\nu+Qp^\nu+Q\gamma^\nu q']I_\nu+2[\gamma^\nu p^\mu+Qg_{\mu\nu}(1-(4pq'/p_3^2))]I_{\mu\nu}-\gamma^\nu F_\nu\}, \tag{C4}$$

where

$$I_{(0;\nu;\mu\nu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu;k_\mu k_\nu)d^4k}{(k^2-2pk)(k^2-2q'\cdot k)(k-p_3)^2k^2} \tag{C5}$$

and

$$F_{(0,\nu)}=\frac{i}{(2\pi)^2}\int\frac{(1;k_\nu)d^4k}{(k^2-2pk)(k^2-2q'\cdot k)(k-p_3)^2}. \tag{C6}$$

To obtain Eq. (B4) we have made use of:

$$I_{\mu\nu}=I_{\nu\mu}$$

and

$$J_3^\lambda=-(\alpha/\pi)(8(pq')/p_3^2)Qg_{\mu\nu}I_{\mu\nu},$$

where $J_3=J_3^0+J_3^\lambda$.

Making use of the results of Appendix A, Sec. C, we get

$$\gamma^\nu I_\nu\doteq im\alpha_1+(Q/2)\alpha_2,$$
$$p^\nu I_\nu=\tfrac{1}{2}(H_0-G_0^{(\mu)}),$$
$$Q\gamma^\nu q'I_\nu\doteq 2im(\mu^2+pq')\alpha_1+Q[(m^2+2pq')\alpha_1-\mu^2\alpha_2+(p_3^2/2)\alpha_3].$$

α_1, α_2, α_3 are defined by

$$I_\nu=\alpha_1p_\nu+\alpha_2q_\nu'+\alpha_3p_{3\nu}.$$

Furthermore, if we let $F_\nu=\eta_1p_\nu+\eta_2q_\nu'+\eta_3p_{3\nu}$ and $G_\nu^{(\mu)}=g_1^{(\mu)}q_\nu'+g_2^{(\mu)}p_{3\nu}$, then it is possible to show that

$$\gamma^\nu p^\mu I_{\nu\mu}\doteq(im/2)\eta_1+(Q/4)(\eta_2-g_1^{(\mu)}).$$

Substituting all this into (C4) gives

$$J_3^0\doteq(\alpha/\pi)\{4im(\mu^2+2pq')\alpha_1-Q[[4(pq')/p_3^2](p_3^2I_0-2H_0)+2G_0^{(m)}-2(\mu^2+2pq')\alpha_2-\tfrac{1}{2}g_1^{(\mu)}]\}, \tag{C7}$$

where we have eliminated α_1 and α_3 in the coefficient of Q by means of the relation $\Delta_{ij}\alpha_j=f_i$. This relation follows from Eq. (A26) where the quantities Δ_{ij} and f_i are defined.

It is easy to see that J_2^0 can be obtained from J_3^0 by an interchange $q\leftrightarrow -q'$. We therefore have

$$J_2^0\doteq(\alpha/\pi)\{4im(\mu^2-2pq)\hat{\alpha}_1-Q[(4pq/p_3^2)(p_3^2\hat{I}_0-2\hat{H}_0)-2G_0^{(m)}+2(\mu^2-2pq)\hat{\alpha}_2+\tfrac{1}{2}g_1^{(\mu)}]\}, \tag{C8}$$

where the "hooded" quantities are to be obtained from the same "unhooded" quantities by the interchange $q\leftrightarrow -q'$.

From Eq. (II.18) we can immediately write

$$J_4=(\alpha/\pi)\{4imG_0^{(m)}+4\gamma^\nu G_\nu^{(m)}\}$$

or by Eq. (A22)

$$J_4\doteq(\alpha/\pi)\{4im[G_0^{(m)}+g_1^{(m)}]\}, \tag{C9}$$

where

$$g_1^{(m)}=[\kappa/(\kappa+2)][G_0^{(m)}+(1/4\kappa m^2)\ln 2\kappa].$$

Collecting terms from Eqs. (C7)–(C9) we get

$$A=4im[(\mu^2+2pq')\alpha_1+(\mu^2-2pq)\hat{\alpha}_1+G_0^{(m)}+g_1^{(m)}],$$
$$-B=(4pq'/p_3^2)(p_3^2I_0-2H_0)+(4pq/p_3^2)(p_3^2\hat{I}_0-2\hat{H}_0)-2(\mu^2+2pq')\alpha_2+2(\mu^2-2pq)\hat{\alpha}_2.$$

α_1 and α_2 are evaluated and listed in Eqs. (A28) and (A29). We have already mentioned that $\hat{\alpha}_1$ and $\hat{\alpha}_2$ can then be obtained from α_1 and α_2. Hence, after some straightforward but nonetheless tedious algebraic manipulations, it can be shown that Eq. (C3) yields

$$\delta_J=-4\left\{[p_3^2G_0^{(m)}+2pq'(p_3^2I_0-2H_0)][1+(p_3^2(\mu^2+2pq')/T_0)]-[p_3^2G_0^{(m)}-2pq(p_3^2\hat{I}_0-2\hat{H}_0)]\times[1+(p_3^2(\mu^2-2pq)/T_0)]+\frac{2(pQ)(p_3^2+2\mu^2)p_3^2G_0^{(\mu)}}{T_0}+\frac{4m^2(pQ)p_3^2(G_0^{(m)}+g_1^{(m)})}{T_0}\right\}. \tag{C10}$$

We have made use of the easily derivable result

$$D=\hat{D}=(m^2/2)(p_3^2+4\mu^2)-\tfrac{1}{4}T_0\,,$$

where D is defined in Eq. (A27).

Equation (C10) is exact. If we now neglect terms of order $m/\mathcal{E}'$ compared with unity we get

$$\delta_J=\left[1+\frac{p_3^2(\mu^2+2pq')}{T_0}\right]\left[\tfrac{1}{2}\ln^2\left(\frac{(2pq')^2}{\mu^2p_3^2}\right)+2\Phi\left(\frac{\mu^2-2pq'}{\mu^2}\right)+\frac{2\pi^2}{3}\right]$$
$$-\left[1+\frac{p_3^2(\mu^2-2pq)}{T_0}\right]\left[\tfrac{1}{2}\ln^2\left(\frac{(2pq)^2}{\mu^2p_3^2}\right)+2\Phi\left(\frac{\mu^2-2pq}{\mu^2}\right)-\frac{4\pi^2}{3}\right]$$
$$-\frac{2(pQ)p_3^2}{T_0}\left(\frac{\kappa'+1}{a'}\right)\left[\ln 2\kappa'\ln\frac{a'-\kappa'}{a'+\kappa'}+\Phi(a'-\kappa')-\Phi(-a'-\kappa')-\pi^2\right],\quad \text{(C11)}$$

where $\kappa'=p_3^2/2\mu^2$ and $a'=(\kappa'^2+2\kappa')^{1/2}$. To obtain Eq. (C11) we have: ignored the last term of Eq. (C10) since it is smaller by a factor $m/\mathcal{E}'$ compared to what is kept; used Eqs. (A40) and (A44) for $(p_3^2I_0-2H_0)$ and $(p_3^2\hat{I}_0-2\hat{H}_0)$, respectively; used Eq. (A19) for $p_3^2G_0^{(\mu)}$; and finally for $p_3^2G_0^{(m)}$ we have approximated the result of Eq. (A20) by

$$p_3^2G_0^{(m)}=-\tfrac{1}{4}[\ln^2(p_3^2/m^2)+(2\pi^2/3)].$$

$\Phi(x)$ is the Spence function defined in Eq. (A49). Since for small x, $\Phi(x)\simeq x$, we have consistently thrown away $\Phi(m/\mathcal{E}')$ terms.

APPENDIX D: SOFT PHOTON INTEGRALS

The basic integral required for the evaluation of δ_s (III.16) is

$$I(p_a,p_b)=\frac{1}{4\pi}\int_0^{\bar{\omega}}\frac{k^2dk}{[k^2+\lambda^2]^{1/2}}\int\frac{(p_ap_b)d\Omega_k}{(p_ak)(p_bk)}+\tfrac{1}{2}K(p_a,p_b)\,.\quad \text{(D1)}$$

We have separated the anticipated infrared term $K(p_a,p_b)$, defined in Eq. (II.10), and consequently $I(p_a,p_b)$ is divergentless.

We first consider the simplest case, i.e., $p_a=p_b=p$. Then Eq. (D1) becomes

$$I(p,p)=-\int_0^{\bar{\omega}}\frac{k^2dk}{[k^2+\lambda^2]^{3/2}}+\ln\frac{m}{\lambda}=1+\ln\frac{m}{2\bar{\omega}}\,.\quad \text{(D2)}$$

Next consider the case $p_a\neq p$; $p_b=p$. Then the first term is

$$I(p,p_a)-\tfrac{1}{2}K(p,p_a)=\frac{1}{2v_a}\int_0^{\bar{\omega}}\frac{kdk}{\omega^2}\ln\left(\frac{1-v_a(k/\omega)}{1+v_a(k/\omega)}\right),$$

where $v_a=|\mathbf{p}_a|/\mathcal{E}_a$. If we let $z=k/\omega=k/[k^2+\lambda^2]^{1/2}$, then

$$I(p,p_a)-\tfrac{1}{2}K(p,p_a)=\frac{1}{2v_a}\int_0^{\bar{z}}\frac{zdz}{1-z^2}\ln\left(\frac{1-v_az}{1+v_az}\right)$$
$$=\frac{1}{4v_a}\int_{-\bar{z}}^{\bar{z}}\frac{dz}{1-z}\ln\left(\frac{1-v_az}{1+v_az}\right),$$

$$I(p,p_a)-\tfrac{1}{2}K(p,p_a)=\frac{1}{4v_a}\left[\ln\left(\frac{1+\bar{z}}{1-\bar{z}}\right)\ln\left(\frac{1-v_a}{1+v_a}\right)+\Phi\left(\frac{2v_a}{1+v_a}\right)-\Phi\left(\frac{-2v_a}{1-v_a}\right)\right],$$

where $\bar{z}=\bar{\omega}/[\bar{\omega}^2+\lambda^2]^{1/2}$ and in performing the last step we have let $\lambda\to 0$ wherever possible. We now make the high energy approximation, $\mathcal{E}_a\gg m_a$, hence $\mathcal{E}_a\simeq|\mathbf{p}_a|+(m_a^2/2\mathcal{E}_a)$. Then

$$I(p,p_a)-\tfrac{1}{2}K(p,p_a)=\frac{1}{4}\left\{\ln\left(\frac{2\bar{\omega}}{\lambda}\right)^2\ln\left(\frac{m_a}{2\mathcal{E}_a}\right)+\Phi(1)-\Phi\left[-\left(\frac{2\mathcal{E}_a}{m_a}\right)^2\right]\right\}.$$

By arguments completely equivalent to those of Appendix A, Sec. D we can show

$$K(p,p_a)=\ln^2\left(\frac{2\mathcal{E}_a}{m_a}\right)-\ln\frac{2\mathcal{E}_a}{m_a}\ln\left(\frac{\lambda}{m}\right)^2+\Phi\left(1-\frac{2m\mathcal{E}_a}{m_a^2}\right).$$

Combining these results and making use of Eqs. (A53) and (A55) we get

$$I(p,p_a)\simeq\ln\frac{\bar{\omega}}{\mathcal{E}_a}\ln\frac{m_a}{2\mathcal{E}_a}-\tfrac{1}{4}\ln^2\left(\frac{m}{2\mathcal{E}_a}\right)+\tfrac{1}{2}\Phi(1)-\tfrac{1}{2}\Phi\left(1-\frac{m_a^2}{2m\mathcal{E}_a}\right).\quad \text{(D3)}$$

Finally we consider $p_a \neq p \neq p_b$. It is shown in Appendix C of Ref. 13 that

$$I(p_a,p_b)=(p_a,p_b)\int_0^1 \frac{dy}{p_y^2}\left[\ln\left(\frac{p_{y0}}{\bar{\omega}}\right)+\tfrac{1}{2}G_{ab}\right],$$

where $p_y=p_a y+p_b(1-y)$ and

$$G_{ab}=\frac{p_{y0}-|\mathbf{p}_y|}{|\mathbf{p}_y|}\ln\left(\frac{p_{y0}+|\mathbf{p}_y|}{p_{y0}-|\mathbf{p}_y|}\right)+2\ln\left(\frac{p_{y0}+|\mathbf{p}_y|}{2p_{y0}}\right).$$

This is exact. We now make some approximations. Both p_a and p_b are highly relativistic and both point in approximately the forward direction. Therefore p_y is also relativistic, i.e., $(p_{y0}-|\mathbf{p}_y|)/p_{y0}$ is of the order $m_a m_b/\mathcal{E}_a\mathcal{E}_b \ll 1$. This shows that we can safely neglect G_{ab} in the above. Therefore

$$I(p_a,p_b)=(p_a p_b)\int_0^1 \frac{dy}{p_y^2}\ln\left(\frac{p_{y0}}{\bar{\omega}}\right). \tag{D4}$$

The y integration is trivial for all but the following three cases:

$$p_a=q \text{ or } q';\ p_b=p' \quad\text{and}\quad p_a=q;\ p_b=q'.$$

In the first two cases p_y^2 has the form ay^2+by-m^2. Hence if we rewrite Eq. (D4) as

$$I(p_a,p_b)=(p_a p_b)\left[\mu(p_a,p_b)\ln\frac{\mathcal{E}_b}{\bar{\omega}}+\int_0^1\frac{\ln(1+cy)}{p_y^2}dy\right], \tag{D5}$$

where $\mu(p_a,p_b)$ is evaluated in Eq. (A48) and $c=(\mathcal{E}_a-\mathcal{E}_b)/\mathcal{E}_b$, we can approximate p_y^2 by ay^2+by in the second term. That is, the contribution to $I(p_a,p')$ coming from small y is almost completely contained in the first term. The error introduced by this approximation can be estimated by the difference in the integrands at $y=0$ times the distance over which the denominator is comparable to m^2, i.e.,

$$\frac{(p'\cdot p_a)c}{b}\times\frac{m^2}{b}\simeq\frac{m}{\text{energy}}$$

and is seen to be small. We may write

$$\int_0^1\frac{dy\ln(1+cy)}{ay^2+by}=\frac{1}{b}\left[\int_0^1\frac{\ln(1+cy)}{y}dy-\int_0^1\frac{\ln(1+cy)}{(y+b/a)}dy\right], \tag{D6}$$

which can be expressed in terms of Spence functions by employing Eq. (D8).

In the last case, where $p_a=q$; $p_b=q'$, we have

$$p_y^2=-\mu^2[2\kappa' y(1-y)+1],\quad\text{where}\quad \kappa'=p_3^2/2\mu^2.$$

The change of variables to $z=2y-1$ then converts Eq. (D4) into

$$I(q,q')=-\left(\frac{\kappa'+1}{\kappa'}\right)\int_{-1}^1\frac{dz}{z^2-(a'/\kappa')^2}\ln\left[\frac{(2E-\mathcal{E}')+\mathcal{E}'z}{2\bar{\omega}}\right]$$

and $a'=(\kappa'^2+2\kappa')^{1/2}$. We rewrite this as

$$I(q,q')=\frac{\kappa'+1}{2a'}\left\{\ln\frac{2E-\mathcal{E}'}{2\bar{\omega}}\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right)^2+\int_{-1}^1\left[\frac{1}{z+(a'/\kappa')}-\frac{1}{z-(a'/\kappa')}\right]\times\ln\left[1+\frac{\mathcal{E}'z}{2E-\mathcal{E}'}\right]dz\right\}. \tag{D7}$$

This last integration can be expressed in terms of Spence function by means of Eq. (D8).

We need to evaluate the following integral:

$$\int_0^1\frac{\ln(1+cy)}{y+r}dy=\int_r^{1+r}\frac{\ln(1-cr)+cy}{y}dy.$$

We have let $y\to y-r$. If we further let $y=[(1-cr)/c]z$ we get

$$\int_0^1\frac{\ln(1+cy)}{y+r}dy=\ln|1-cr|\ln\left(\frac{1+r}{r}\right)-\Phi\left(\frac{(1+r)c}{cr-1}\right)+\Phi\left(\frac{rc}{cr-1}\right). \tag{D8}$$

APPENDIX E: HARD PHOTON INTEGRALS

In this section we evaluate the integrals which appear in Eq. (III.51)

$$\mathcal{I}_\alpha^{(i)}=\frac{(p_3^2)^2}{2T_0}\int_{\bar{\omega}}^{\Delta E}\omega d\omega\int_{E-|q|}^{\bar{y}}\frac{dy}{|\mathbf{q}-\mathbf{k}|}\int_0^{2\pi}\frac{d\varphi}{2\pi}A_\alpha^{(i)} \tag{E1}$$

for all the $A_\alpha^{(i)}$ listed in Eqs. (III.29)–(III.49). y is the same as in Eq. (III.14) and is defined by

$$y=-q\cdot k/\omega=E-|\mathbf{q}|\cos\phi, \tag{E2}$$

and $\bar{y}$ is given in Eq. (III.15). Throughout we shall assume that

$$\bar{y}\gg m \quad\text{and}\quad \mathcal{E}'+\Delta E<\mathcal{E}_{\max}'. \tag{E3}$$

These conditions, as can be seen from the discussion following Eq. (III.12), are equivalent to Eq. (III.12) and, in fact, have already been invoked in the derivation of Eq. (E1). Furthermore, in evaluating Eq. (E1) it shall be our practice to neglect terms of order unity.

In order to perform the φ integrations it is first necessary to explicitly exhibit the φ dependence of $A_\alpha^{(i)}$. To achieve this the following relations, derivable directly from the geometry of Fig. 3, will be used:

$$p'\cdot k=-\omega(u-v\cos\varphi)\,, \tag{E4}$$

$$q'\cdot k=-\omega(a+v\cos\varphi)\,, \tag{E5}$$

$$q_3{}^2=b+c\cos\varphi\,, \tag{E6}$$

where

$$\begin{aligned} a&=y+m-u\,,\\ b&=p_3{}^2+2m\omega-2\omega u\,,\\ c&=2\omega v\,, \end{aligned} \tag{E7}$$

$$\begin{aligned} u&=\mathcal{E}'-|\mathbf{p}'|\cos\theta\cos(\phi+\eta)\,,\\ v&=|\mathbf{p}'|\sin\theta\sin(\phi+\eta)\,. \end{aligned} \tag{E8}$$

We can express u and v in terms of y by means of Eq. (III.7) and the relation

$$\cos(\phi+\eta)=(E-\omega-y)/(|\mathbf{q}-\mathbf{k}|)\,,$$

which again follows directly from the geometry of Fig. 3. The following table of integrals can be seen to have relevance to our problems:

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{a'+b'\cos y}{a+b\cos y}dy=\frac{b'}{b}+\frac{a'-a(b'/b)}{[a^2-b^2]^{1/2}}\,, \tag{E9}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{a'+b'\cos y}{(a+b\cos y)^2}dy=\frac{aa'-bb'}{[a^2-b^2]^{3/2}}\,, \tag{E10}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dy}{(a+b\cos y)^2(a'+b'\cos y)}=\frac{ab}{(ba'-ab')[a^2-b^2]^{3/2}}-\frac{b'}{(ba'-ab')^2}\left[\frac{b}{[a^2-b^2]^{1/2}}-\frac{b'}{[a'^2-b'^2]^{1/2}}\right], \tag{E11}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dy}{(a+b\cos y)^2(a'+b'\cos y)^2}=\frac{1}{(ba'-ab')^2}\left[\frac{b^2a}{(a^2-b^2)^{3/2}}+\frac{b'^2a'}{(a'^2-b'^2)^{3/2}}\right]-\frac{2bb'}{(ba'-ab')^3}\left[\frac{b}{(a^2-b^2)^{1/2}}-\frac{b'}{(a'^2-b'^2)^{1/2}}\right]. \tag{E12}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dy}{(a+b\cos y)(a'+b'\cos y)(a''+b''\cos y)}=\frac{b^2}{(a'b-b'a)(a''b-b''a)(a^2-b^2)^{1/2}}+\frac{b'^2}{(ab'-ba')(a''b'-b''a')(a'^2-b'^2)^{1/2}}+\frac{b''^2}{(ab''-ba'')(a'b''-b'a'')(a''^2-b''^2)^{1/2}}. \tag{E13}$$

From this table one can already see the advantage of defining

$$D_{p'}=(\mathbf{q}-\mathbf{k})^2(u^2-v^2);\quad D_{q'}=(\mathbf{q}-\mathbf{k})^2(a^2-v^2);\quad D=(\mathbf{q}-\mathbf{k})^2(b^2-c^2)\,.$$

$D_{p'}$, $D_{q'}$, and D are, of course, functions of y.

Some straightforward manipulations then yield

$$D_{p'}(y)=[m(E+m)-(y+m)(\mathcal{E}'+\omega)]^2+m^2[\mathbf{q}^2-(E-y)^2]\,, \tag{E14}$$

$$D_{q'}(y)=[(E-\mathcal{E}'+m)^2-\mu^2]y^2+2(\mathcal{E}'-m)[\mu^2-m(E-\mathcal{E}'+m)]y+m(\mathcal{E}'-m)[2\mu^2+m(\mathcal{E}'-m)]\,, \tag{E15}$$

$$D(y)=4\{[m(\mathcal{E}'-m)|\mathbf{q}|-\omega y(\mathcal{E}'+\omega-m)]^2+2\omega m(\mathcal{E}'-m)(\mathcal{E}'+\omega-m)(E-|\mathbf{q}|)(E+|\mathbf{q}|-y)\}\,. \tag{E16}$$

After having performed the φ integrations the resulting integrands of Eq. (E1) are algebraic functions of y. That is to say, the y integrations are never that complex so as not to be found in any standard table of integrals. However, the performance of the y integrations leads to unwieldy expressions and the problem then is to reduce these expressions into a manageable form. The following properties of $D_{p'}(y)$ and $D_{q'}(y)$ aid in performing this reduction:

$D_{p'}(y)$, $\quad D_{q'}(y)$,

has a relative minimum for y equal to

$$y_{p'}=\frac{m(E-\mathcal{E}'-\omega)}{\mathcal{E}'+\omega+m}, \qquad y_{q'}=-m+\frac{mE(E-\mathcal{E}')-\mu^2\mathcal{E}'}{(E-\mathcal{E}')^2},$$

and the value of the function at this minimum is

$$m^2\left[\mathbf{q}^2-\frac{(E+m)^2(\mathcal{E}'+\omega-m)}{\mathcal{E}'+\omega+m}\right] \qquad m^2q^2-\frac{[(E+m)E(\mathcal{E}'-m)-\mathbf{q}^2\mathcal{E}']^2}{\mathcal{E}'^2+\mathbf{q}^2-m^2-2(\mathcal{E}'-m)(E+m)},$$

which is approximately equal to

$$[m^2/(\mathcal{E}'+\omega)][2mE(E-\mathcal{E}'-\omega)-\mu^2(\mathcal{E}'+\omega)], \qquad [\mu^2\mathcal{E}'/(E-\mathcal{E}')^2][2mE(E-\mathcal{E}')-\mu^2\mathcal{E}'].$$

At $y=E-|\mathbf{q}|$ the function has the value

$$[m(E+m)-(E-|\mathbf{q}|+m)(\mathcal{E}'+\omega)]^2, \qquad [(E-|\mathbf{q}|+m)(|\mathbf{q}|+\mathcal{E}')-m(E+m)]^2.$$

By expanding about the minimum

$$D_{p'}(y)=D_{p'}(y_{p'})+(y-y_{p'})^2(\mathcal{E}'+\omega)^2, \qquad D_{q'}(y)=D_{q'}(y_{q'})+(y-y_{q'})^2(E-\mathcal{E}')^2,$$

we can approximate the value of the function at $y=\bar{y}$ as

$$D_{p'}(\bar{y})=[\bar{y}(\mathcal{E}'+\omega)]^2, \qquad D_{q'}(y)=[\bar{y}(E-\mathcal{E}')]^2,$$

where we have made use of Eq. (E3) in the last step.

To illustrate how these properties are used in a typical y integration, consider

$$\int_{E-|\mathbf{q}|}\frac{dy}{[D_{q'}(y)]^{1/2}}=\frac{1}{c^{1/2}}\ln\left[[D_{q'}(y)]^{1/2}+c^{1/2}y+\frac{b}{2c^{1/2}}\right]\Bigg|_{E-|\mathbf{q}|}^{\bar{y}}$$

where b and c are defined by $D_{q'}(y)=a+by+cy^2$. Then $-b/2c=y_{q'}$ and we write the argument of the log function as $[D_{q'}(y)]^{1/2}+c^{1/2}(y-y_{q'})$. At the upper limit this is approximately $2\bar{y}(E-\mathcal{E}')$, where we have used $c^{1/2}\simeq(E-\mathcal{E}')$, and at the lower limit we have

$$(E-|\mathbf{q}|+m)(E+\mathcal{E}')-mE+(E-\mathcal{E}')\left[E-|\mathbf{q}|+m-\frac{mE(E-\mathcal{E}')-\mu^2\mathcal{E}'}{(E-\mathcal{E}')^2}\right]=2(E-|\mathbf{q}|)E+\frac{\mu^2\mathcal{E}'}{E-\mathcal{E}'}=\frac{\mu^2E}{E-\mathcal{E}'}.$$

Therefore

$$\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{[D_{q'}(y)]^{1/2}}\simeq\frac{1}{E-\mathcal{E}'}\ln\frac{2(E-\mathcal{E}')^2\bar{y}}{\mu^2E}=\frac{1}{E-\mathcal{E}'}\ln\left[\left[\frac{2(E-\mathcal{E}')}{\mu}\right]^2\frac{\bar{\omega}}{\omega}\right].$$

We are now prepared to tackle the integration of Eq. (E1). For $i=1$ we have:

(1a) $\alpha=q^2$. $A_{q^2}{}^{(1)}$ is independent of φ and

$$\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{y^2[\mathbf{q}^2+\omega^2-2\omega E+2\omega y]^{1/2}}=\frac{1}{E-\omega}\left[\frac{1}{E-|\mathbf{q}|}-\frac{1}{\bar{y}}\right]\simeq\frac{2E}{\mu^2(E-\omega)}.$$

Therefore,

$$\mathcal{J}_{q^2}{}^{(1)}=-\frac{E}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{[T_0+4m\omega(p_3{}^2+4pq+2m\omega)]d\omega}{\omega(E-\omega)}$$

or

$$\mathcal{J}_{q^2}{}^{(1)}=\ln(\bar{\omega}/\Delta E)+O(\Delta E/E). \tag{E17}$$

(1b) $\alpha=q'^2$. Employing Eq. (E10) (with $a'=1$ and $b'=0$) for the φ integration we get

$$\mathcal{J}_{q'^2}{}^{(1)}=-\frac{\mu^2}{2}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{\omega}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{n(y)}{[D_{q'}(y)]^{3/2}}dy,$$

where $D_{q'}(y)$ is given in Eq. (E15) and

$$n(y)=(\mathbf{q}-\mathbf{k})^2a =\omega y^2+[\mathbf{q}^2-\omega E-(\mathcal{E}'-m)(E+\omega+m)]y +(\mathcal{E}'-m)[\mu^2+mE-\omega m].$$

The y integration although cumbersome is straightforward. It yields (upon neglecting 1 compared to $\bar{y}/m$)

$$\int_{E-|\mathbf{q}|}^{\bar{y}} n(y)\frac{dy}{[D_{q'}(y)]^{3/2}}=\frac{2}{\mu^2}\left(1-\frac{\omega}{E-\mathcal{E}'}\right).$$

Therefore

$$\mathcal{J}_{q'^2}{}^{(1)}=\ln(\bar{\omega}/\Delta E)+O[\Delta E/(E-\mathcal{E}')]. \quad \text{(E18)}$$

(1c) $\alpha=qq'$. This time Eq. (E9) is relevant to the φ integration. The required y integration is then

$$\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{y[D_{q'}(y)]^{1/2}}=\frac{1}{\mu^2a'}\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right)^2,$$

where $a'=(\kappa'^2+2\kappa')^{1/2}$ and $\kappa'=p_3{}^2/2\mu^2$. This result again neglects m compared to $\bar{y}$. Finally we find that

$$\mathcal{J}_{qq'}{}^{(1)}=\frac{-2(\kappa'+1)}{a'}\left[\ln\frac{\bar{\omega}}{\Delta E}\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right) +O\left(\frac{\Delta E}{E}\ln\left(\frac{a'+\kappa'}{a'-\kappa'}\right)\right)\right]. \quad \text{(E19)}$$

(1*d*) $\alpha=q$; $\alpha=q'$; $\alpha=1$. The pattern is clear so we simply state the results:

$$\mathcal{J}_q{}^{(1)}=-\left(\frac{\Delta E}{E}\right)\frac{2p_3{}^2(\mu^2+mE)}{T_0}\ln\left(\left(\frac{2E}{\mu}\right)^2\frac{\bar{\omega}}{\Delta E}\right), \quad \text{(E20)}$$

$$\mathcal{J}_{q'}{}^{(1)}=\left(\frac{\Delta E}{E-\mathcal{E}'}\right)\frac{p_3{}^2(p_3{}^2+2\mu^2-2mE)}{T_0} \times\ln\left(\left[\frac{2(E-\mathcal{E}')}{\mu}\right]^2\frac{\bar{\omega}}{\Delta E}\right), \quad \text{(E21)}$$

$$\mathcal{J}_1{}^{(1)}=(\Delta E/E)[\mathcal{E}'/2(E-\mathcal{E}')].$$

These can be seen to be of order unity and are negligible in our calculation.

It is clear from the preceding that if we expand the integrands of $\mathcal{J}_\alpha{}^{(1)}$ as $(1/\omega)[a+b(\omega/E)+c(\omega/E)^2+\cdots]$, then only the first term contributes to the integral; the rest are of order $\Delta E/E$. That is, we could have neglected k everywhere except in the denominator terms $(q\cdot k)$ and $(q'\cdot k)$. Moreover, in these leading terms the y integration never depends on the upper limit $\bar{y}$. That is, we could have taken as our region of integration the isotropic region of radius ΔE. But this is precisely the soft photon approximation, where now ΔE replaces $\bar{\omega}$. We therefore see that the soft photon approximation is valid for all photons emitted by the heavier particle. It then follows that the net effect of adding $\mathcal{J}_\alpha{}^{(1)}$ to the corresponding terms in the soft photon cross section should be to replace $\bar{\omega}$ in those terms by ΔE. This is so.

We next turn our attention to $i=2$.

(2a) $\alpha=p^2$. Upon employing Eq. (E10) for the φ integration, there results

$$\mathcal{J}_{p^2}{}^{(2)}=\frac{(p_3{}^2)^2}{2T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{\omega}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{(\mathbf{q}-\mathbf{k})^2(ba'-cb')}{[D(y)]^{3/2}}dy,$$

where $D(y)$ is given in Eq. (E16), b and c are defined by Eq. (E8) and

$$a'=-T_0-4\omega y(p_3{}^2-4mE+2\omega y)-4\omega u(\mu^2+2mE-2\omega y),$$
$$b'=4\omega v(\mu^2+2mE-2\omega y).$$

The essential contribution to this integral comes from small values of ω. This is true for two reasons: First the $1/\omega$ term which appears explicitly in the integrand, and second since $D(y)$ has no sharp minimum in the range of y integration the result of this integration is something nearly proportional to $\bar{y}$ (actually to $\bar{y}-E+|\mathbf{q}|$), i.e., to $1/\omega$. The second reason is more important in that it has application to later integrals. Then in seeking the leading terms of the numerator $(\mathbf{q}-\mathbf{k})^2(ba'-cb')$, we treat ω as negligible compared to a typical energy and ωy as comparable to mE. This leads to

$$(\mathbf{q}-\mathbf{k})^2(ba'-cb')=-16\mathcal{E}'(E-\mathcal{E}')(mE-\omega y)^2\omega(\bar{y}-y).$$

In the same spirit we can rewrite Eq. (E16) as

$$D(y)\simeq[2\mathcal{E}'(mE-\omega y)]^2. \quad \text{(E22)}$$

Then if we let $z=\omega y$ and $\bar{z}=\omega\bar{y}=T_0/8m(E-\mathcal{E}')$, we get

$$\mathcal{J}_{p^2}{}^{(2)}=-\frac{m}{2\bar{z}}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{\omega^2}\int_{\omega(E-|\mathbf{q}|)}^{\bar{z}}\frac{\bar{z}-z}{mE-z}dz.$$

We do this integration and get

$$\mathcal{J}_{p^2}{}^{(2)}=-\frac{m}{2\bar{\omega}}\left[1+\frac{4m\mu^2\mathcal{E}'}{T_0}\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right].$$

A moment's consideration shows that the soft photon approximation is poorest when applied to $I(p,p)$, the term corresponding to $\mathcal{J}_{p^2}{}^{(2)}$. To improve this we can let ϵ replace $\bar{\omega}$ in Eq. (III.18), where $\epsilon\ll\bar{\omega}$ and add to $\mathcal{J}_{p^2}{}^{(2)}$ the term

$$-\frac{m}{2\bar{z}}\int_\epsilon^{\bar{\omega}}\frac{d\omega}{\omega}\int_{E-|\mathbf{q}|}^{E+|\mathbf{q}|}\frac{\bar{z}-\omega y}{mE-\omega y}dy.$$

The net result of performing these operations is found to be identical with the result obtained by simply replacing, by 1, the coefficient $m/2\bar{\omega}$ in the above expression for $\mathcal{J}_{p^2}{}^{(2)}$. We therefore write

$$\mathcal{J}_{p^2}{}^{(2)}=-\left[1+\frac{4m\mu^2\mathcal{E}'}{T_0}\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right]. \quad \text{(E23)}$$

This can be seen to be of order unity and is negligible in our calculation.

(2b) $\alpha=p'^2$. With the aid of Eq. (E12) we can show that $\mathcal{I}_{p'^2}{}^{(2)}=(\mathrm{I})+(\mathrm{II})+(\mathrm{III})+(\mathrm{IV})$, where

$$(\mathrm{I})=-\frac{m^2\mathcal{E}'^2}{2T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega[T_0-4m\omega(\mu^2+2mE)]}{\omega(\mathcal{E}'+\omega)^2}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{(\mathbf{q}-\mathbf{k})^2u}{[D_{p'}(y)]^{3/2}}dy,$$

$$(\mathrm{II})=-\frac{m\mathcal{E}'^2}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega[T_0-4m\omega(\mu^2+2mE)]}{(\mathcal{E}'+\omega)^3}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{[D_{p'}(y)]^{1/2}},$$

$$(\mathrm{III})=-\frac{4m^2\mathcal{E}'^2}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{\omega d\omega[T_0-4m\omega(\mu^2+2mE)]}{(\mathcal{E}'+\omega)^2}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{[m(\mathcal{E}'+\omega)-\omega u](\mathbf{q}-\mathbf{k})^2}{[D(y)]^{3/2}}dy,$$

$$(\mathrm{IV})=-\frac{2m\mathcal{E}'^2}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{\omega d\omega[T_0-4m\omega(\mu^2+2mE)]}{(\mathcal{E}'+\omega)^3}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{[D(y)]^{1/2}}.$$

The y integration in (I) is arduous. It can, however, be shown that

$$\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{(\mathbf{q}-\mathbf{k})^2u}{[D_{p'}(y)]^{3/2}}dy\simeq\frac{2\mathcal{E}'}{m^2(\mathcal{E}')+\omega}.$$

$D_{p'}(y)$ has a sharp minimum at

$$y=m(E-\mathcal{E}'-\omega)/(\mathcal{E}'+\omega+m)$$

and the above result depends on the condition that the y integration region contains this minimum. This condition is identical with the condition (E3) which is assumed to obtain. Furthermore, (II), (III), and (IV) are all of the order $m/\mathcal{E}'$. This will become more apparent in the next case (2c) where the y integrations of (II), (III), (IV) are done. That (I) is the only surviving term could have been guessed at from the start. For if we assume that the essential contribution to $\mathcal{I}_{p'}{}^{(2)}$ comes from $\mathbf{k}\|\mathbf{p}'$ [because of the $(p'\cdot k)^2$ in the denominator] then in $q_3{}^2$ we can let $k=\omega p'/\mathcal{E}'$, i.e.,

$$q_3{}^2=p_3{}^2-2k(p-p')=p_3{}^2[(\mathcal{E}'+\omega)/\mathcal{E}'].$$

This leads directly to (I). Therefore

$$\mathcal{I}_{p'^2}{}^{(2)}=-\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{\omega}\left(\frac{\mathcal{E}'}{\mathcal{E}'+\omega}\right)^3\left[1-\frac{4m\omega(\mu^2+2mE)}{T_0}\right]$$

and neglecting terms of order unity we get

$$\mathcal{I}_{p'^2}{}^{(2)}\simeq\ln\bar{\omega}/\Delta E. \tag{E24}$$

(2c) $\alpha=pp'$. Applying Eq. (E11) to do the φ integration we get

$$\mathcal{I}_{pp'}{}^{(2)}=\frac{\mathcal{E}'^3}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{(\mathcal{E}'+\omega)^2\omega}(A+B),$$

where

$$A=\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{N(y)}{[D_{p'}(y)]^{1/2}}dy,$$

$$B=\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{2N(y)}{[D(y)]^{1/2}}\left[1+\frac{4m(\mathcal{E}'+\omega)(\mathbf{q}-\mathbf{k})^2(m(\mathcal{E}'+\omega)-\omega u)}{D(y)}\right]\omega dy, \tag{E25}$$

$$N(y)=T_0+2\omega y(p_3{}^2-4mE+2\omega y).$$

It can be shown that

$$A=\frac{N(y_0)}{\mathcal{E}'+\omega}\ln\left(\frac{2(\mathcal{E}'+\omega)^2\bar{y}}{m^2E}\right)+\frac{2\omega\bar{y}}{\mathcal{E}'+\omega}[\omega\bar{y}-2m(2E-\mathcal{E}')],$$

where $y_0=m(E-\mathcal{E}'-\omega)/(\mathcal{E}'+\omega+m)$. To evaluate B we first pick out the leading terms in a manner analogous to that used in $\mathcal{I}_{p^2}{}^{(2)}$ [case (2a)]. Then

$$B=\int_{\omega(E-|\mathbf{q}|)}^{\bar{z}}\frac{T_0+2z(p_3{}^2-4mE+2z)}{\mathcal{E}'(mE-z)}\left[1+\frac{mE}{mE-z}\right]dz,$$

which integrates to

$$B=\frac{1}{\mathcal{E}'+\omega}\left\{\frac{[T_0-4m^2E(E-\mathcal{E}')]\bar{z}}{mE-\bar{z}}+(4m^2E^2-T_0)\ln\left(1-\frac{\bar{z}}{mE}\right)+2\bar{z}[2m(2E-\mathcal{E}')-\bar{z}]\right\}.$$

The last terms of A and B cancel completely. With the aid of the relations

$$\frac{1}{\omega}\left(\frac{\mathcal{E}'}{\mathcal{E}'+\omega}\right)^3=\frac{1}{\omega}-\frac{1}{\mathcal{E}'+\omega}-\frac{\mathcal{E}'}{(\mathcal{E}'+\omega)^2}-\frac{\mathcal{E}'^2}{(\mathcal{E}'+\omega)^3},$$

$$y_0(m\mathcal{E}'-2mE+\omega y_0)=m^2(\mathcal{E}'+\omega)\left[1-\frac{2\mathcal{E}'}{\mathcal{E}'+\omega}-\frac{E(E-3\mathcal{E}')}{(\mathcal{E}'+\omega)^2}-\frac{E^2\mathcal{E}'}{(\mathcal{E}'+\omega)^3}\right],$$

we can do the remaining integration. We finally get

$$\mathcal{I}_{pp'}{}^{(2)}=2\ln\frac{\Delta E}{\bar{\omega}}\ln\frac{2\mathcal{E}'}{m}-\tfrac{1}{2}\ln^2\frac{\Delta E}{\bar{\omega}}+R\ln\xi-\Phi\left(\frac{-\Delta E}{\mathcal{E}'}\right)+\left(\frac{4m^2\mathcal{E}'^2}{T_0}-1\right)I_1-\left(\frac{4m^2\mathcal{E}'^2}{T_0}+\frac{1}{2}\right)I_2-\frac{4m^2E(E-3\mathcal{E}')}{3T_0}I_3$$
$$-\frac{m^2E^2}{T_0}I_4+\left[\frac{T_0-4m^2E(E-\mathcal{E}')}{4m\mu^2\mathcal{E}'}+\left(\frac{4m^2E^2}{T_0}-1\right)\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right]\ln\frac{\Delta E}{\bar{\omega}},\qquad\text{(E26)}$$

where

$$\xi=\mathcal{E}'/(\mathcal{E}'+\Delta E);\quad R=2\ln(2\mathcal{E}'/m)-\ln(\Delta E/\bar{\omega})-\ln\xi;$$
$$I_1=(1-\xi)(R+2)+\xi\ln\xi,$$
$$I_2=(1-\xi^2)(R+1)+\xi^2\ln\xi+(1-\xi),$$
$$I_3=(1-\xi^3)(R+\tfrac{2}{3})+\xi^3\ln\xi+(1-\xi)+\tfrac{1}{2}(1-\xi^2),$$
$$I_4=(1-\xi^4)(R+\tfrac{1}{2})+\xi^4\ln\xi+(1-\xi)+\tfrac{1}{2}(1-\xi^2)+\tfrac{1}{3}(1-\xi^3).$$

The last term of Eq. (E26) comes from B; the rest from A.

(2d) $\alpha=p$. This case is not very different than case (2a) and analogous considerations lead to

$$\mathcal{I}_p{}^{(2)}=-\frac{2m}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{\omega}\int_{\omega(E-|\mathbf{q}|)}^{z}\frac{m(z)dz}{(mE-z)^2},$$

where

$$m(z)=2(E-\mathcal{E}')z^2-(4mE^2-6mE\mathcal{E}'-\mu^2\mathcal{E}')z+2mE(2mE^2-3mE\mathcal{E}'-\mu^2\mathcal{E}').$$

Upon performing this integration we get

$$\mathcal{I}_p{}^{(2)}=-\left[\frac{mE(E-\mathcal{E}')}{\mu^2\mathcal{E}'}+\frac{p_3{}^2(\mu^2+2mE)}{T_0}\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right]\ln\left(\frac{\Delta E}{\bar{\omega}}\right).\qquad\text{(E27)}$$

(2e) $\alpha=p'$. Following the same procedure as in case (2c) we can show that

$$\mathcal{I}_{p'}{}^{(2)}=\frac{\mathcal{E}'^2}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{(\mathcal{E}'+\omega)^2}(A'+B'),$$

where A' and B' are the same as A and B (E25) with $N(y)$ replaced by $M(y)$ and

$$M(y)=T_0-4m^2\mathcal{E}'(3E-\mathcal{E}')+2m\omega[4m\mathcal{E}'-\mu^2-6mE+2m\omega]+4\omega y[3m\mathcal{E}'-2mE+2m\omega+\omega y].$$

However, since in this case we have one less power of ω in the denominator compared to case (2c), we need only consider the leading term, i.e., the term corresponding to the first term in A. Using the expansion

$$\frac{\mathcal{E}'^2M(y_0)}{(\mathcal{E}'+\omega)^3T_0}=\frac{-p_3{}^2(\mu^2+2mE+2m\mathcal{E}')}{T_0}\frac{\mathcal{E}'}{(\mathcal{E}'+\omega)^2}+\left(\frac{1}{2}+\frac{8m^2\mathcal{E}'^2}{T_0}\right)\frac{\mathcal{E}'^2}{(\mathcal{E}'+\omega)^3}-\frac{12m^2E\mathcal{E}'}{T_0}\frac{\mathcal{E}'^3}{(\mathcal{E}'+\omega)^4}+\frac{4m^2E^2}{T_0}\frac{\mathcal{E}'^4}{(\mathcal{E}'+\omega)^5},$$

we get

$$\mathcal{J}_{p'}{}^{(2)}=\frac{-p_3{}^2(\mu^2+2mE+2m\mathcal{E}')}{T_0}I_1+\left(\frac{4m^2\mathcal{E}'^2}{T_0}+\frac{1}{4}\right)I_2-\frac{4m^2E\mathcal{E}'}{T_0}I_3+\frac{m^2E^2}{T_0}I_4\,, \tag{E28}$$

where I_1, I_2, I_3, I_4 are given below Eq. (E26).

(2f) $\alpha=1$. It can be shown that

$$\mathcal{J}_1{}^{(2)}=\frac{\Delta E}{E}\left[\frac{mE(E-\mathcal{E}')}{\mu^2\mathcal{E}'}-\frac{4m^2E(3E-\mathcal{E}')}{T_0}\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right].$$

This is of order unity and negligible. The reasoning used in the previous case to allow us to disregard all but the leading term is applicable here to anticipate this result.

Finally, when $i=3$ we have:

(3a) $\alpha=pq$. We use Eq. (E9) to perform the φ integration and get

$$\mathcal{J}_{pq}{}^{(3)}=\frac{p_3{}^2E}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{\omega}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{y}\left[\frac{\mu^2+2mE}{(\mathbf{q}^2+\omega^2-2\omega E+2\omega y)^{1/2}}-\frac{T_0+(p_3{}^2+2m\omega)(\mu^2+2mE)}{[D(y)]^{1/2}}\right].$$

The y integration, neglecting terms of order $\Delta E/E$, yields

$$-\frac{T_0}{p_3{}^2E}\ln\left(\frac{2E}{\mu}\right)^2\left(\frac{\bar{\omega}}{\omega}\right)+\frac{T_0+p_3{}^2(\mu^2+2mE)}{p_3{}^2E}\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right).$$

Therefore,

$$\mathcal{J}_{pq}{}^{(3)}=-\ln\frac{\Delta E}{\bar{\omega}}\left[\ln\left(\frac{2E}{\mu}\right)^2-\tfrac{1}{2}\ln\frac{\Delta E}{\bar{\omega}}-\left(1+\frac{p_3{}^2(\mu^2+2mE)}{T_0}\right)\ln\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right]. \tag{E29}$$

(3b) $\alpha=p'q'$. Upon doing the φ integration with the aid of Eq. (E13) it is found that

$$\mathcal{J}_{p'q'}{}^{(3)}=-\frac{mE\mathcal{E}'}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{[T_0-2m\omega(\mu^2+2mE)]}{(\mathcal{E}'+\omega)\omega}(Y_1+Y_2+Y_3)d\omega\,,$$

where

$$\begin{aligned}
Y_1&=\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{(y+m)[D_{p'}(y)]^{1/2}}=\frac{1}{mE}\ln\left(\frac{(2E)^2}{\mu^2+2mE}\right),\\
Y_2&=\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{(y+m)[D_{q'}(y)]^{1/2}}=\frac{1}{mE}\ln\left(\frac{\mu^2-2mE}{\mu^2}\right),\\
Y_3&=\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{\omega dy}{[(m\mathcal{E}')-\omega y]}\left[\frac{1}{[D_{q'}(y)]^{1/2}}-\frac{2\omega}{[D(y)]^{1/2}}\right].
\end{aligned} \tag{E30}$$

From Eq. (E15) and Eq. (E16) we can show that $(2\omega)^2D_{q'}(y)=D(y)$ for $y=m\mathcal{E}'/\omega$. Moreover, since $D_{q'}(y)$ has a sharp minimum at $y\simeq -m$, we can safely make the estimate

$$\int\frac{dy}{(m\mathcal{E}'-\omega y)[D_{q'}(y)]^{1/2}}=\frac{1}{[D_{q'}(m\mathcal{E}'/\omega)]^{1/2}}\int\frac{dy}{m\mathcal{E}'-\omega y}+\frac{1}{m(\mathcal{E}'+\omega)}\int\frac{dy}{[D_{q'}(y)]^{1/2}}.$$

Furthermore, from Eq. (E22), it follows that

$$\int\frac{dy}{(m\mathcal{E}'-\omega y)[D(y)]^{1/2}}=\frac{1}{[D(m\mathcal{E}'/\omega)]^{1/2}}\int\frac{dy}{(m\mathcal{E}'-\omega y)}-\frac{1}{m(E-\mathcal{E}')}\int\frac{dy}{[D(y)]^{1/2}}.$$

Therefore,

$$\begin{aligned}
Y_3&=\frac{\omega}{m}\left[\frac{1}{\mathcal{E}'+\omega}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{[D_{q'}(y)]^{1/2}}+\frac{2\omega}{E-\mathcal{E}'}\int_{E-|\mathbf{q}|}^{\bar{y}}\frac{dy}{[D(y)]^{1/2}}\right]\\
&=\frac{\omega}{m(E-\mathcal{E}')(\mathcal{E}'+\omega)}\left[2\ln\frac{2(E-\mathcal{E}')}{\mu}-\ln\frac{\omega}{\bar{\omega}}-\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right].
\end{aligned}$$

It can be shown that the contribution to $\mathcal{G}_{p'q'}^{(3)}$ arising from this term is approximately

$$-\frac{E}{E-\mathcal{E}'}\left[2\ln\left(\frac{2(E-\mathcal{E}')}{\mu}\right)-\ln\frac{\Delta E}{\bar{\omega}}\right]\left[(1-\xi)+\frac{p_3^2(\mu^2+2mE)}{T_0}(1-\xi+\ln\xi)\right],$$

which in spite of the large logarithmic terms is still of order unity and hence negligible. From Y_1 and Y_2 then we get

$$\mathcal{G}_{p'q'}^{(3)}=-2\ln\frac{2E}{\mu}\left[\ln\frac{\Delta E}{\bar{\omega}}+\left(1+\frac{p_3^2(\mu^2+2mE)}{T_0}\right)\ln\xi\right]. \qquad \text{(E31)}$$

(3c) $\alpha=pq'$. Equation (E13), with $a''=1$ and $b''=0$, is used to do the φ integration. We get

$$\mathcal{G}_{pq'}^{(3)}=\frac{m\mathcal{E}'(E-\mathcal{E}')}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega}{\omega}\int_{E-|q|}^{\bar{y}}\frac{T(y)dy}{(m\mathcal{E}'-\omega y)}\left[\frac{1}{[D_{q'}(y)]^{1/2}}-\frac{2\omega}{[D(y)]^{1/2}}\right],$$

where $T(y)=T_0+2\omega y(p_3^2+\mu^2-2mE)$. If we now exploit the strong resemblance borne by the integrand in Y_3, Eq. (E30), of the previous case to the integrand here, we can immediately write for the y integration

$$\frac{1}{m}\left[\frac{1}{\mathcal{E}'+\omega}\int_{E-|q|}^{\bar{y}}\frac{T(y)dy}{[D_{q'}(y)]^{1/2}}+\frac{2\omega}{E-\mathcal{E}'}\int_{E-|q|}^{\bar{y}}\frac{T(y)dy}{[D(y)]^{1/2}}\right.$$
$$=\frac{1}{m(\mathcal{E}'+\omega)(E-\mathcal{E}')}\left[T(y_0)\ln\left[\left(\frac{2(E-\mathcal{E}')}{\mu}\right)^2\frac{\bar{\omega}}{\omega}\right]-T\left(\frac{mE}{\omega}\right)\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right],$$

where y_0 represents that value of y for which $D_{q'}(y)$ has a minimum and $y_0\simeq-m$. Consequently,

$$\mathcal{G}_{pq'}^{(3)}=2\ln\frac{\Delta E}{\bar{\omega}}\ln\frac{2(E-\mathcal{E}')}{\mu}-\tfrac{1}{2}\ln^2\frac{\Delta E}{\bar{\omega}}-\left(1+\frac{2mE(p_3^2+\mu^2-2mE)}{T_0}\right)\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\ln\frac{\Delta E}{\bar{\omega}}. \qquad \text{(E32)}$$

In this result we have neglected the term

$$\left[1+\frac{p_3^2(p_3^2+\mu^2-2mE)}{T_0}\right]\ln\xi\left[2\ln\left(\frac{2(E-\mathcal{E}')}{\mu}\right)-\ln\left(\frac{\Delta E}{\bar{\omega}}\right)\right],$$

which is of the same order as the contribution to $\mathcal{G}_{p'q'}^{(3)}$ [in case (3b)] arising from Y_3, i.e., of order unity.

(3d) $\alpha=p'q$. Here, as in the last case, Eq. (E13) is again applicable to the integration $d\varphi$. Accordingly we get

$$\mathcal{G}_{p'q}^{(3)}=\frac{m\mathcal{E}'}{T_0}\int_{\bar{\omega}}^{\Delta E}\frac{d\omega(E-\mathcal{E}'-\omega)}{\omega(\mathcal{E}'+\omega)}[T_0+2m\omega(p_3^2-6mE-\mu^2+2m\omega)][D_1+D_2],$$

where

$$D_1=\int_{E-|q|}^{\bar{y}}\frac{dy}{y[D_{p'}(y)]^{1/2}};\quad D_2=\int_{E-|q|}^{\bar{y}}\frac{2\omega dy}{y[D(y)]^{1/2}}.$$

By using Eq. (E22) for $D(y)$ it is easy to show that

$$D_2=\frac{\omega}{mE\mathcal{E}'}\left[\ln\left[\left(\frac{2E}{\mu}\right)^2\frac{\bar{\omega}}{\omega}\right]-\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right)\right]$$

and therefore its contribution to $\mathcal{G}_{p'q}^{(3)}$ is given approximately by

$$-\ln\xi[2\ln(2E/\mu)-\ln(\Delta E/\bar{\omega})],$$

which we neglect. D_1 is more complicated. However, if we neglect m compared to $\bar{y}$ it can be reduced to

$$D_1=[2/m(E-\mathcal{E}'-\omega)]\ln[2(E-\mathcal{E}'-\omega)/\mu].$$

The ω integration then yields

$$\mathcal{G}_{p'q}^{(3)}=2\left\{\ln\frac{\Delta E}{\bar{\omega}}\ln\frac{2(E-\mathcal{E}')}{\mu}+\left[1+\frac{p_3^2(\mu^2+6mE)}{T_0}\right]\ln\xi\ln\frac{2E}{\mu}\right\}. \qquad \text{(E33)}$$

(3e) $\alpha=p$. Equation (E9) serves to do the φ integration. We next require

$$\int_{E-|\mathbf{q}|}^{\bar{y}}\left\{\frac{p_3{}^2(\mu^2+2mE)+2mE(2mE-\mu^2+2m\omega)-2p_3{}^2\omega y}{[D(y)]^{1/2}}-\frac{2mE}{|\mathbf{q}-\mathbf{k}|}\right\}dy$$

$$=-\frac{p_3{}^2\mu^2+2mE(2mE-\mu^2+2m\omega)}{T_0}\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right).$$

This leads directly to the result

$$\mathcal{J}_p{}^{(3)}=-\left(\frac{\mu^2p_3{}^2+2mE(2mE-\mu^2)}{T_0}\right)\ln\frac{\Delta E}{\bar{\omega}}\ln\left(\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')}\right). \quad \text{(E34)}$$

(3f) $\alpha=p'$. The φ and resulting y integrations are analogous to case (3d), whence we get (neglecting order unity)

$$\mathcal{J}_{p'}{}^{(3)}=-\frac{\mathcal{E}'}{T_0}\int_{\omega}^{\Delta E}\frac{d\omega}{(\mathcal{E}'+\omega)^2}P(y_0)\left[2\ln\frac{2(\mathcal{E}'+\omega)}{m}-\ln\frac{\omega}{\bar{\omega}}\right],$$

where

$$P(y_0)=p_3{}^2(p_3{}^2-4mE)+2m\omega(3p_3{}^2-6mE+4m\omega)+2\omega[m(E-\mathcal{E}'-\omega)/(\mathcal{E}'+\omega)](p_3{}^2+2m\omega).$$

With the aid of the result

$$P(y_0)/(\mathcal{E}'+\omega)^2=(2m)^2-[8m^2E/(\mathcal{E}'+\omega)],$$

we get

$$\mathcal{J}_{p'}{}^{(3)}=-\left[\frac{4mEp_3{}^2}{T_0}\left[R\ln\xi+\Phi\left(-\frac{\Delta E}{\mathcal{E}'}\right)\right]+\frac{(p_3{}^2)^2}{T_0}\frac{\Delta E}{\mathcal{E}'}(R-1-\ln\xi)\right]. \quad \text{(E35)}$$

(3g) $\alpha=q$; $\alpha=q'$; $\alpha=1$. Since these cases resemble closely enough some one of the preceding cases so as to make the technique for their evaluation self-evident (and their contributions are anyway negligible), we simply state results:

$$\mathcal{J}_1{}^{(3)}=\frac{\Delta E}{E}\frac{12m^2E^2}{T_0}\ln\frac{\mu^2\mathcal{E}'}{2mE(E-\mathcal{E}')},$$

$$\mathcal{J}_q{}^{(3)}=\frac{\Delta E}{E}\left[1+\frac{(4mE-p_3{}^2)(4mE-\mu^2)}{T_0}\right]\left[2\ln\frac{2E}{\mu}-\ln\frac{\Delta E}{\bar{\omega}}\right],$$

$$\mathcal{J}_{q'}{}^{(3)}=-\frac{\Delta E}{E-\mathcal{E}'}\left[\frac{(2mE-p_3{}^2)^2-2\mu^2(3mE-p_3{}^2)}{T_0}\right]\left[2\ln\frac{2(E-\mathcal{E}')}{\mu}-\ln\frac{\Delta E}{\bar{\omega}}\right].$$